

1. Let $G = \langle a \rangle \cong C_{36}$, $N = \langle a^{24} \rangle$, and $g = a^{16}$.

a) What is the order of g in G , and of $\bar{g} = Ng$ in $\bar{G} = G/N$?

b) What is the number of elements h outside of N for which $o(h) = o(\bar{h})$?

Solution: a) $o(g) = o(a^{16}) = \frac{o(a)}{\gcd(o(a), 16)} = \frac{36}{\gcd(36, 16)} = 9$, The order of $\bar{g}$ is the smallest positive k such that $g^k \in N$, where $N = \{1, a^{24}, a^{12}\}$. Since $g = a^{16} \notin N$, $g^2 = a^{32} \notin N$ and $g^3 = a^{48} = a^{12} \in N$, we get that $o(\bar{g}) = 3$.

Another way to calculate $o(\bar{g})$ would use the fact that $G = \langle a \rangle$ is cyclic, so $\langle \bar{a} \rangle = \bar{G}$ is also cyclic of order $|G|/|N| = 12$ (since $|N| = o(a^{24}) = \frac{36}{\gcd(36, 24)} = 3$). Thus for $\bar{g} = \bar{a}^{16} = \bar{a}^{16}$, the order is $o(\bar{g}) = \frac{12}{\gcd(12, 16)} = 3$.

b) $o(h) = o(\bar{h})$ if and only if $h^k \notin N$ for any $0 < k < o(h)$ (clearly, $h^{o(h)} = 1 \in N$), that is, if $\langle h \rangle \cap N = 1$. This clearly holds for those elements h whose order is coprime to $|N|$, since then $|\langle h \rangle \cap N|$ divides $\gcd(o(h), |N|) = 1$. On the other hand, if $1 < d = \gcd(o(h), |N|)$, then $H = \langle h \rangle$ and N both being cyclic (subgroups of a cyclic group), they both contain a subgroup of order d , but G is cyclic, so it contains only one such subgroup. This implies that $N \cap H \neq 1$.

In our special case, $o(h)$ must be coprime to 3, and divide 36, so it is a divisor of 4. Consequently, h is a nontrivial element of the unique subgroup of order 4 of G , that is, there are 3 such elements h .

2. Let G be a group and $H \leq G$. Are the following statements true or false?

- There always exists a group L and homomorphism $\varphi : G \rightarrow L$ such that $\text{Ker } \varphi = H$.
- If $H \triangleleft G$, then there exists a group L and homomorphism $\varphi : G \rightarrow L$ such that $\text{Ker } \varphi = H$.
- There always exists a group L and homomorphism $\varphi : L \rightarrow G$ such that $\text{Im } \varphi = H$.
- If $\varphi : G \rightarrow L$ is a homomorphism then $\varphi(H) \leq L$.
- If $H \triangleleft G$, and $\varphi : G \rightarrow L$ is a homomorphism then $\varphi(H) \triangleleft L$.
- If $H \triangleleft G$, and $\varphi : G \rightarrow L$ is a homomorphism then $\varphi(H) \triangleleft \varphi(G)$.
- If $\varphi : G \rightarrow L$ is a homomorphism then $|G| \mid |\varphi(G)|$.
- If $\varphi : G \rightarrow L$ is a homomorphism then $|\varphi(G)| \mid |G|$.
- A group L is a homomorphic image of G (that is, there exists a surjective homomorphism $G \rightarrow L$) if and only if L is isomorphic to a factor group of G .

Solution: a) False. The kernel must be a normal subgroup.

b) True, $\varphi : G \rightarrow G/H$, $\varphi(g) = Hg$ is such a homomorphism.

c) True, $\varphi : H \rightarrow G$, $\varphi(h) = h$ is such a homomorphism.

d) True because $1 = \varphi(1) \in \varphi(H)$, and for $h, h' \in H$ we have $\varphi(h)\varphi(h')^{-1} = \varphi(h(h')^{-1}) \in \varphi(H)$.

e) False. If $S \leq L$ is not a normal subgroup then for the homomorphism $\varphi : S \rightarrow L$, $\varphi(s) = s$ it is true that $S \triangleleft S$ but $\varphi(S) = S$ is not a normal subgroup in L .

f) True. We have seen in part d) that $\varphi(H)$ is a subgroup in L (and it is clearly contained in $\varphi(G)$), furthermore, for any $\varphi(g) \in \varphi(G)$ and $\varphi(h) \in \varphi(H)$ (where $h \in H$) $\varphi(g)^{-1}\varphi(h)\varphi(g) = \varphi(g^{-1}hg) \in \varphi(H)$ because $g^{-1}hg \in H$.

g) False, for example, if $|G| \neq 1$, and $\varphi = 1$ then $|\varphi(G)| = 1$ is not divisible by $|G|$.

h) True because $\varphi(G) = \text{Im } \varphi \cong G / \text{Ker } \varphi$, so $|\varphi(G)| = \frac{|G|}{|\text{Ker } \varphi|} \mid |G|$.

- i) True. The homomorphism theorem ensures that if $\varphi : G \rightarrow L$ is surjective then $G/\text{Ker } \varphi \cong \text{Im } \varphi = L$. Conversely, if $\varphi : G/N \rightarrow L$ is an isomorphism for some $N \triangleleft G$ then with the natural homomorphism $\psi : G \rightarrow G/N, g \mapsto Ng$, the image of $\varphi \circ \psi$ is L .

3. Let $H \leq G$ and $M, N \triangleleft G$. Prove that

- $H \cap N \triangleleft H; N \cap M \triangleleft G;$
- $HN \leq G; NM \triangleleft G;$
- if $M \leq N$ then G/N is a homomorphic image of G/M .

Solution: a) The intersection of subgroups is a subgroup so we only have to prove that they are closed under conjugation.

$x \in H \cap N, h \in H \Rightarrow h^{-1}xh \in H$ because $h, x \in H$, and $h^{-1}xh \in N$ because $x \in N \triangleleft G$, so $h^{-1}xh \in H \cap N$.

For $x \in N \cap M, g \in G, g^{-1}xg \in N$ and $\in M$, since N and M are both normal, so $g^{-1}xg \in N \cap M$.

- b) $N \triangleleft G \Rightarrow HN = NH \Rightarrow HN \leq G$.

From this it also follows that $NM \leq G$. Furthermore, NM is closed under conjugation by elements of G , since for $g \in G, n \in N, m \in M$ we have $g^{-1}nmg = (g^{-1}ng)(g^{-1}mg) \in NM$.

- c) By the 2nd isomorphism theorem, $(G/M)/(N/M) \cong G/N$, so G/N is a factor group of G/M , and then by problem 1.i), it is also a homomorphic image of G/M .

4. Let $N \triangleleft G$. Prove that the map $H \mapsto H/N := \{Nh \mid h \in H\}$ is a bijection between the subgroups of G containing N and the subgroups of G/N . Show that this map connects normal subgroups with normal subgroups, and it preserves the inclusion of subgroups, that is, $H_1 \leq H_2$ if and only if the subgroup of G/N corresponding to H_1 is contained in the one corresponding to H_2 .

Solution: Let $S(G)$ and $S(G/N)$ be the set of subgroups of G and G/N , respectively, and $S(G)_{\geq N}$ the set of those subgroups of G which contain N . Let

$$\Phi : S(G) \rightarrow S(G/N), \quad \Phi(H) = \{Nh \mid h \in H\}$$

$(\Phi(H))$ is indeed a subgroup, since $N1 = N$ is the identity element of G/N , $Nh_1Nh_2 = Nh_1h_2$, and $(Nh)^{-1} = Nh^{-1}$. Conversely,

$$\Psi : S(G/N) \rightarrow S(G), \quad \Psi(\{Ng_i \mid i \in I\}) = \bigcup_{i \in I} Ng_i$$

(this also forms a subgroup, because $1 \in N$, and for $a \in Ng_i$ and $b \in Ng_j$, we have $ab \in Ng_iNg_j$ and $a^{-1} \in (Ng_i)^{-1}$, where $\bar{1} = N$, Ng_iNg_j and $(Ng_i)^{-1}$ are in the given subgroup of G/N).

Note that the elements of $\text{Im } \Psi$ are subgroups containing N since $\bar{1} = N$ is in every subgroup of G/N .

We show that the restriction of Φ to $S(G)_{\geq N}$ (denote it by Φ_1) and Ψ are inverses of each other.

If $N \leq H \leq G$ then $\Psi(\Phi_1(H)) = \bigcup \{Nh \mid h \in H\} \subseteq H$, and every element of H is in the

union, since $h = 1h \in Nh$, so $\Psi(\Phi_1(H)) = H$.

Conversely, if $\mathcal{H} = \{Ng_i \mid i \in I\} \in S(G/N)$, then $\Phi_1(\Psi(\mathcal{H})) = \{Nx \mid x \in Ng_i \text{ for some } i \in I\}$. But for these elements x we have $Nx = Ng_i$, so $\Phi_1(\Psi(\mathcal{H})) = \mathcal{H}$.

We have shown that Φ_1 (and the same way Ψ) is a bijection between $S(G)_{\geq N}$ and $S(G/N)$. It is clear that Φ_1 (and Φ , too) and Ψ preserves inclusion, that is, if $H_1 \leq H_2$, then $\Phi(H_1) \leq \Phi(H_2)$, and if $\mathcal{H}_1 \leq \mathcal{H}_2$, then $\Psi(\mathcal{H}_1) \leq \Psi(\mathcal{H}_2)$.

If $H \triangleleft G$, then $\Phi(H)$ is also closed under conjugation: $(Ng)^{-1}(Nh)(Ng) = Ng^{-1}hg \in \Phi(H)$ because $g^{-1}hg \in H$ for every $g \in G$. Conversely, if $\mathcal{H} \triangleleft G/N$, then for every $g \in G$ and $a \in Ng_i \in \mathcal{H}$, we have $g^{-1}ag \in (Ng)^{-1}Ng_iNg \in \mathcal{H}$.

5. Prove that if $N \triangleleft G$, and $|G : N|$ is even then there is an H with $N \leq H \leq G$ such that $|H : N| = 2$.

Solution: $|G/N|$ is even, so it has an element of order 2 (see problem 3/2, or the Cauchy theorem, which we are going to prove later on in the lecture), and this generates a subgroup $\mathcal{H}$ of order 2. The map Ψ of the previous problem assigns a subgroup H to $\mathcal{H}$ which is the union of two cosets of N ($N \cup Nt$ for $t \in G \setminus N$), so $N \leq H$, and $|H : N| = 2$.

6. Let $N \triangleleft G$, $H \leq G$, $|G| = 24$, $|N| = 4$, and $|H| = 6$. What can be the cardinality of H at the natural homomorphism $G \rightarrow G/N$? Give an example for each case.

Solution: The image of H by the natural map $G \rightarrow G/N$ is $NH/N \cong H/(N \cap H)$, so its cardinality is $|H|/|N \cap H|$. Since $|N \cap H|$ is a divisor of $|N| = 4$ and of $|H| = 6$, it divides $\gcd(4, 6) = 2$, hence $|N \cap H| = 1$ or 2 . Thus the cardinality of the image of H is either 6 or 3.

Both cases are possible. If $G = S_4$, $N = \langle 1, (..)(..) \rangle$ is the Klein group and $H = S_3 = \langle (12)(123) \rangle$ then $N \cap H = 1$, so the image of H has cardinality 6. If $G = \langle a \rangle \cong C_{24}$, $N = \langle a^6 \rangle$ and $H = \langle a^4 \rangle$ then $N \cap H = \langle a^{12} \rangle \cong C_2$, so the image of H has cardinality 3.

7. What is the number of elements of order 4 in A_8 ?

Solution: In the disjoint cycle decomposition of an element g of order 4, there is at least one 4-cycle, and apart from the 4-cycles, there are only 2-cycles and fixed-points. If $g \in A_n$ then g has evenly many cycles of even length, so it either contains two 4-cycles or one 4-cycle and one 2-cycle. Of the first type, there are $\binom{8}{4}(3!)^2 \cdot \frac{1}{2} = 1260$ elements in A_8 , of the second, there are $\binom{8}{4}3!\binom{4}{2} = 2520$ elements, so A_8 contains altogether 3780 elements of order 4.

8. Prove that the elements (123) and (234) of A_4 are conjugate in S_4 but they are not conjugate in A_4 .

Solution: Let $g = (123) = (123)(4)$ and $h = (234) = (234)(1)$. The two elements have the same cycle structure, so they are conjugate in S_4 .

The element h can be written in three different ways, keeping the order of the cycle lengths aligned with those of g : we can permute the elements in the 3-cycle cyclically. We can determine the conjugating elements of S_4 according to the given form of h .

$$\begin{array}{ccc} (123)(4) & (123)(4) & (123)(4) \\ (234)(1) & (342)(1) & (423)(1) \end{array}$$

The corresponding conjugating elements (that map every element of the base set to the one below it) are (1234) , (1324) , and (14) , respectively, and neither of them are in A_4 , so g and h are not conjugate in A_4 .

- HW1.** Let $G = \langle a \rangle \cong C_{24}$ and $N = \langle a^{40} \rangle$. Determine the order of the (normal) subgroup N and of the factor group G/N . Find an element $g \in G$ such that $o(\bar{g}) = 4$ in G/N but $o(g) \neq 4$ in G .
- HW2.** Suppose that $N \triangleleft G$, $H \leq G$, $|G| = 100$, $|N| = 20$ and $|H| > 20$. Prove that there is a subgroup $L \leq H$ with $|H : L| = 5$. (Hint: what can be the order of NH ?)