Perturbation theory of random graphs

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Outline

- Blown-up matrices burdened by Wigner-noise have as many structural eigenvalues as the rank of the blown-up matrix.
- Recovering the blown-up structure and relation to generalized quasirandom graphs and Szemerédi's Regularity Lemma.
- Convergent graph sequences, graphons, testable graph parameters, Lovász, Szegedy, J. Comb. Theory B, 2006.
- Testability of minimum balanced multiway cut densities and relation to statistical physics.

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Notation

Definition

The $n \times n$ symmetric real matrix $\mathbf{W}$ is a Wigner-noise if its entries w_{ij}, $1 \leq i \leq j \leq n$, are independent random variables, $\mathbb{E}w_{ij} = 0$, $\mathtt{Var}\, w_{ij} \leq \sigma^2$ with some $0 < \sigma < \infty$ and the w_{ij}'s are uniformly bounded (there is a constant $K > 0$ such that $|w_{ij}| \leq K$).

Füredi, Komlós (Combinatorica, 1981):

$$\max_{1 \leq i \leq n} |\lambda_i(\mathbf{W})| \leq 2\sigma\sqrt{n} + O(n^{1/3}\log n)$$

with probability tending to 1 as $n \to \infty$.

Sharp concentration theorem

Theorem

W *is an* $n \times n$ *real symmetric matrix, its entries in and above the main diagonal are independent random variables with absolute value at most 1.* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$*: eigenvalues of* **W**.
For any $t > 0$*:*

$$\mathbb{P}\left(|\lambda_i - \mathbb{E}(\lambda_i)| > t\right) \leq \exp\left(-\frac{(1-o(1))t^2}{32i^2}\right) \quad \textit{when} \quad i \leq \frac{n}{2},$$

and the same estimate holds for the probability

$$\mathbb{P}\left(|\lambda_{n-i+1} - \mathbb{E}(\lambda_{n-i+1})| > t\right).$$

Alon, Krivelevich, Vu, Israel J. Math., 2002

Previous results imply:

Lemma

There exist positive constants C_1 and C_2, depending on the common bound K for the entries of the Wigner-noise $\mathbf{W}$, such that

$$\mathbb{P}\left(\|\mathbf{W}\| > C_1 \cdot \sqrt{n} \right) \leq \exp(-C_2 \cdot n).$$

Borel–Cantelli Lemma $\Longrightarrow$
The spectral norm of $\mathbf{W}$ is $\mathcal{O}(\sqrt{n})$ almost surely.

Perturbation results for weighted graphs

$\mathbf{A} = \mathbf{B} + \mathbf{W}$, where
$\mathbf{W}$: $n \times n$ Wigner-noise
$\mathbf{B}$: $n \times n$ blown-up matrix of $\mathbf{P}$ with blow-up sizes $n_1, \dots, n_k$, $\sum_{i=1}^{k} n_i = n$.
$\mathbf{P}$: $k \times k$ pattern matrix
k is kept fixed as $n_1, \dots, n_k \to \infty$ "at the same rate": there is a constant c such that
$\frac{n_i}{n} \geq c, \quad i = 1, \dots k$.
growth rate condition: g.r.c.

Spectrum of a noisy graph

$G_n = (V, \mathbf{A})$, $\mathbf{A} = \mathbf{B} + \mathbf{W}$ is $n \times n$, $n \to \infty$
$\mathbf{B}$ induces a planted partition $P_k = (V_1, \dots, V_k)$ of V.
Weyl's perturbation theorem $\Longrightarrow$
Adjacency spectrum of G_n: under g.r.c. there are k structural eigenvalues of order n (in absolute value) and the others are $\mathcal{O}(\sqrt{n})$, almost surely.
The eigenvectors $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_k)$ corresponding to the structural eigenvalues are "not far" from the subspace of stepwise constant vectors on P_k $\Longrightarrow$

$$S_k^2(\mathbf{X}) \le S_k^2(P_k, \mathbf{X}) = \mathcal{O}(\frac{1}{n}), \quad \text{almost surely.}$$

Laplacian spectrum is not so informative.

Spectrum of the normalized Laplacian

$G_n = (V, \mathbf{A})$, $\mathbf{A} = \mathbf{B} + \mathbf{W}$ is $n \times n$, $n \to \infty$
$\mathbf{L}_D = \mathbf{I} - \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}$

Theorem

There exists a positive number $\delta \in (0,1)$, independent of n, such that for every $0 < \tau < 1/2$ the following statement holds with probability tending to 1 as $n \to \infty$, under the g.r.c.:
there are exactly k eigenvalues of $\mathbf{L}_D$ that are located in the union of intervals $[-n^{-\tau}, 1-\delta+n^{-\tau}]$ and $[1+\delta-n^{-\tau}, 2+n^{-\tau}]$, while all the others are in the interval $(1-n^{-\tau}, 1+n^{-\tau})$.

Representation: $\mathbf{x}_i = \mathbf{D}^{-1/2}\mathbf{u}_i, \quad (i = 1, \dots, k)$

$$\tilde{S}_k^2(P_k, \mathbf{X}) \le \frac{k}{(\frac{\delta}{n^{-\tau}} - 1)^2} \quad \text{w. p. to 1 as } n \to \infty, \quad \text{under g.r.c.}$$

Noisy graph is simple with appropriate noise

The uniform bound K on the entries of $\mathbf{W}$ is such that $\mathbf{A} = \mathbf{B} + \mathbf{W}$ has entries in [0,1].
With an appropriate Wigner-noise the noisy matrix $\mathbf{A}$ is a generalized random graph: edges between V_i and V_j exist with probability $0 < p_{ij} < 1$.
For $1 \leq i \leq j \leq k$ and $l \in V_i$, $m \in V_j$:

$$w_{lm} := \begin{cases} 1 - p_{ij}, & \text{with probability} \quad p_{ij} \\ -p_{ij} & \text{with probability} \quad 1 - p_{ij} \end{cases}$$

be independent random variables, otherwise $\mathbf{W}$ is symmetric. The entries have zero expectation and bounded variance:

$$\sigma^2 = \max_{1 \leq i \leq j \leq k} p_{ij}(1 - p_{ij}) \leq \frac{1}{4}.$$

Szemerédi's Regularity Lemma

For any graph on n vertices there exist a partition $(V_0, V_1, \ldots, V_k)$ of the vertices (here V_0 is a "small" exceptional set) such that "most" of the V_i, V_j pairs $(1 \le i < j \le k)$ are ε-regular with $\varepsilon > 0$ fixed in advance.

The pair V_i, V_j $(i \neq j)$ is ε-regular, if for any $A \subset V_i$, $B \subset V_j$ with $|A| > \varepsilon |V_i|$, $|B| > \varepsilon |V_j|$:

$$|\mathrm{dens}\,(A, B) - \mathrm{dens}\,(V_i, V_j)| < \varepsilon,$$

where

$$\mathrm{dens}\,(A, B) = \frac{e(A, B)}{|A| \cdot |B|}$$

is the edge-density between the disjoint vertex-sets A and B.

Informally, ε-regularity means that the edge-densities between the V_i, V_j pairs are homogeneous.

If the graph is sparse, then $k = 1$, otherwise k can be arbitrarily large (but it depends only on ε).

The planted partition is ε-regular almost surely

With the above Wigner-noise, $e(V_i, V_j)$ is the sum of $|V_i| \cdot |V_j|$ independent, identically distributed Bernoulli variables with parameter p_{ij} $(1 \le i, j \le k)$. Hence, $e(A, B)$ is binomially distributed with expectation $|A| \cdot |B| \cdot p_{ij}$ and variance $|A| \cdot |B| \cdot p_{ij}(1 - p_{ij})$.
By Chernoff's inequality for large deviations:

$$\begin{aligned}\mathbb{P}\left(|\mathrm{dens}\,(A, B) - p_{ij}| > \varepsilon\right) &\le e^{-\frac{\varepsilon^2 |A|^2 |B|^2}{2[|A||B|p_{ij}(1-p_{ij}) + \varepsilon |A||B|/3]}} \\ &= e^{-\frac{\varepsilon^2 |A||B|}{2[p_{ij}(1-p_{ij}) + \varepsilon/3]}} \\ &\le e^{-\frac{\varepsilon^4 |V_i||V_j|}{2[p_{ij}(1-p_{ij}) + \varepsilon/3}}\end{aligned}$$

that tends to 0, as $|V_i| = n_i \to \infty$ and $|V_j| = n_j \to \infty$. Hence, any pair V_i, V_j is ε-regular with probability tending to 1 if $n_1, \dots, n_k \to \infty$ under the g.r.c. (weaker than the structure guaranteed by Szemerédi's Lemma)

Recognizing the structure

Theorem

Let $\mathbf{A}_n$ be a sequence of $n \times n$ matrices, where $n \to \infty$. Assume that $\mathbf{A}_n$ has exactly k eigenvalues of order greater than $\sqrt{n}$, and there is a k-partition of the vertices such that the k-variance of the representatives is $\mathcal{O}(\frac{1}{n})$, in the representation with the corresponding eigenvectors. Then there is a blown-up matrix $\mathbf{B}_n$ such that $\mathbf{A}_n = \mathbf{B}_n + \mathbf{E}_n$ with $\|\mathbf{E}_n\| = \mathcal{O}(\sqrt{n})$.

Proof: construction by the cluster centers.
Results with planted partitions and cut-matrices or low-rank approximation of the column space of $\mathbf{A}$:

- Frieze, A., Kannan, R.
- McSherry, F.
- Amin Coja-Oghlan

Edge- and node-weighted graphs

$G = G_n$: weighted graph on the node set $[n] = \{1, \ldots, n\} = V(G)$.
Edge-weights: $\beta_{ij} = \beta_{ji} \in \mathbb{R}$ (strength of the interaction between the nodes).
For randomization purposes suppose that $\beta_{ij} \in [0, 1]$ (0=no edge).
Node-weights: $\alpha_i > 0$, $i = 1, \ldots, n$ (individual weights of the nodes).
Let $\mathcal{G}$ denote the set of such weighted graphs.
$\alpha_G := \sum_{i=1}^{n} \alpha_i$ (volume of G)
$\alpha_U := \sum_{i \in U} \alpha_i$ (volume of the node-set $U \subset V(G)$)

$$e_G(U, T) := \sum_{u \in U} \sum_{t \in T} \alpha_u \alpha_t \beta_{ut}, \qquad U, T \subset V = V(G)$$

$\mathcal{P}_k$: set of k-partitions $P = (V_1, \ldots, V_k)$ of V.

Definition

The homomorphism density between the simple graph F ($|V(F)| = k$) and the weighted graph G:

$$t(F,G) = \frac{1}{(\alpha_G)^k} \sum_{\Phi: V(F)\to V(G)} \prod_{i=1}^{k} \alpha_{\Phi(i)} \prod_{ij\in E(F)} \beta_{\Phi(i)\Phi(j)}$$

For simple G, this is the probability that a random map $F \longrightarrow G$ is a homomorphism.

Definition

The sequence (G_n) is (left) convergent if $t(F, G_n)$ is convergent for any simple graph F.

G_n's become more and more similar in small details.

As most maps into a large graph are injective, we consider mainly injective homomorphisms and use the notation:

$$\alpha_\Phi = \prod_{i=1}^{k} \alpha_{\Phi(i)}, \qquad inj_\Phi(F,G) = \prod_{ij \in E(F)} \beta_{\Phi(i)\Phi(j)},$$

$$ind_\Phi(F,G) = \prod_{ij \in E(F)} \beta_{\Phi(i)\Phi(j)} \prod_{ij \in E(\bar{F})} (1 - \beta_{\Phi(i)\Phi(j)}),$$

$$t_{inj}(F,G) = \frac{1}{(\alpha_G)^k} \sum_{\Phi \; inj.} \alpha_\Phi \cdot inj_\Phi(F,G),$$

$$t_{ind}(F,G) = \sum_{\Phi \; inj.} \alpha_\Phi \cdot ind_\Phi(F,G).$$

Randomization

A simple graph on k vertices is selected at random based on the weighted graph G:
k vertices are chosen with replacement with respective probabilities $\alpha_i(G)/\alpha(G)$ $(i = 1, \dots, n)$. Given the node-set $\{\Phi(1), \dots, \Phi(k)\}$, the edges come into existence conditionally independently, with probabilities of the edge-weights. $\xi(k, G)$ is the resulted random graph.

$$\mathbf{P}(\xi(k, G) = F) \sim t_{ind}(F, G), \quad t(F, G) \sim t_{inj}(F, G) \quad (k \ll n)$$

and there is a well-defined relation between $t_{inj}(F, G)$ and $t_{ind}(F, G)$.

Graphons

Borgs et al. (2006) also construct the limit object: that is a $W : [0,1] \times [0,1] \to \mathbb{R}$ symmetric, bounded, measurable function, graphon

The interval [0,1] corresponds to the vertices and the values $W(x,y) = W(y,x)$ to the edge-weights.

The set of symmetric, measurable functions $W : [0,1] \times [0,1] \to [0,1]$ is denoted by $\mathcal{W}_{[0,1]}$.

The stepfunction graphon $W_G \in \mathcal{W}_{[0,1]}$ is assigned to the weighted graph $G \in \mathcal{G}$ in the following way: the sides of the unit square are divided into intervals $I_1, \dots, I_n$ of lengths $\alpha_1/\alpha_G, \dots, \alpha_n/\alpha_G$, and over the rectangle $I_i \times I_j$ the stepfunction takes on the value β_{ij}.

Cut-distance

The cut-distance between the graphons W and U is

$$\delta_\square(W,U) = \inf_\nu \|W - U^\nu\|_\square$$

where the cut-norm of the graphon W is defined by

$$\|W\|_\square = \sup_{S,T\subset[0,1]} \left| \iint_{S\times T} W(x,y)\,dx\,dy \right|,$$

and the infimum is taken over all measure preserving bijections $\nu : [0,1] \to [0,1]$, while U^ν denotes the transformed U after performing the same measure preserving bijection ν on both sides of the unit square.

An equivalence relation is defined over the set of graphons: two graphons belong to the same class if they can be transformed into each other by a measure preserving map, i.e., their $\delta_\Box$-distance is zero.
By a Theorem of Lovász, Szegedi, 2006: the classes of $\mathcal{W}_{[0,1]}$ form a compact metric space with the $\delta_\Box$ metric.

$$\delta_\Box(G, G') = \delta_\Box(W_G, W_{G'}) \quad \text{and} \quad \delta_\Box(W, G) = \delta_\Box(W, W_G).$$

A sequence of weighted graphs with uniformly bounded edge-weights is convergent if and only if it is a Cauchy sequence in the metric $\delta_\Box$.

Weak Szemerédi Lemma

Lemma

(Borgs et al., 2006) For every ε, every weighted graph G has a partition P into at most $4^{1/\varepsilon^2}$ classes such that

$$\delta_\square(G, G/P) \le \varepsilon \|G\|_2 \le \varepsilon.$$

$$\|G\|_2 = \sqrt{\sum_{i,j} \frac{\alpha_i \alpha_j}{\alpha_G^2} \beta_{ij}^2}$$

$$\alpha_i(G/P) = \frac{\alpha_{V_i}}{\alpha_G}, \quad \beta_{ij}(G/P) = \frac{e_G(V_i, V_j)}{\alpha_{V_i} \alpha_{V_j}}$$

Testability of weighted graph parameters

Definition

A weighted graph parameter f is testable if for every $\varepsilon > 0$ there is a positive integer k such that if $G \in \mathcal{G}$ satisfies

$$\max_i \frac{\alpha_i(G)}{\alpha_G} \leq \frac{1}{k},$$

then

$$\mathbb{P}(|f(G) - f(\xi(k, G))| > \varepsilon) \leq \varepsilon,$$

where $\xi(k, G)$ is a random simple graph on k nodes randomized "appropriately" from G.

Equivalent statements of testability

Theorem

Equivalent statements for the testability of the bounded weighted graph parameter f.

- *For every $\varepsilon > 0$ there is a positive integer k such that for every weighted graph $G \in \mathcal{G}$ satisfying the node-condition $\max_i \alpha_i(G)/\alpha_G \le 1/k$, $|f(G) - \mathbb{E}(f(\xi(k,G)))| \le \varepsilon$.*
- *For every left-convergent weighted graph sequence (G_n) with $\max_i \alpha_i(G_n)/\alpha_{G_n} \to 0$, $f(G_n)$ is also convergent ($n \to \infty$).*
- *f can be extended to graphons such that $\tilde{f}(W)$ is continuous in the cut-norm and $\tilde{f}(W_{G_n}) - f(G_n) \to 0$, whenever $\max_i \alpha_i(G_n)/\alpha_{G_n} \to 0$ ($n \to \infty$).*
- *For every $\varepsilon > 0$ there is an $\varepsilon_0 > 0$ real and an $n_0 > 0$ integer such that if G_1, G_2 are weighted graphs satisfying $\max_i \alpha_i(G_1)/\alpha_{G_1} \le 1/n_0$, $\max_i \alpha_i(G_2)/\alpha_{G_2} \le 1/n_0$, and $\delta_\square(G_1, G_2) < \varepsilon_0$, then $|f(G_1) - f(G_2)| < \varepsilon$.*

Minimum multiway cut densities

Let $k < n$ be a fixed positive integer.

$$f_k(G) := \min_{P \in \mathcal{P}_k} \frac{1}{\alpha_G^2} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} e_G(V_i, V_j)$$

minimum k-way cut density of G.
Let $c \le 1/k$ be a fixed positive real number.
$\mathcal{P}_k^c$: set of k-partitions of V such that $\frac{\alpha_{V_i}}{\alpha_G} \ge c$ $(i = 1, \dots, k)$, or equivalently, $c \le \frac{\alpha_{V_i}}{\alpha_{V_j}} \le \frac{1}{c}$ $(i \ne j)$.

$$f_k^c(G) := \min_{P \in \mathcal{P}_k^c} \frac{1}{\alpha_G^2} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} e_G(V_i, V_j)$$

minimum c-balanced k-way cut density of G.

Let $\mathbf{a} = \{a_1, \dots, a_k\}$ be a probability distribution on $[k]$.
$\mathcal{P}_k^{\mathbf{a}}$: set of k-partitions of V such that

$$\left(\frac{\alpha_{V_1}}{\alpha_G}, \dots, \frac{\alpha_{V_k}}{\alpha_G}\right)$$

is approximately $\mathbf{a}$-distributed.

$$f_k^{\mathbf{a}}(G) := \min_{P \in \mathcal{P}_k^{\mathbf{a}}} \frac{1}{\alpha_G^2} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} e_G(V_i, V_j)$$

minimum $\mathbf{a}$-balanced k-way cut density of G.

Minimum weighted multiway cut densities

We want to penalize extremely different cluster volumes.

$$\mu_k(G) := \min_{P\in\mathcal{P}_k} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \frac{1}{\alpha_{V_i} \cdot \alpha_{V_j}} \cdot e_G(V_i, V_j)$$

minimum weighted k-way cut density of G.

$$\mu_k^c(G) := \min_{P\in\mathcal{P}_k^c} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \frac{1}{\alpha_{V_i} \cdot \alpha_{V_j}} \cdot e_G(V_i, V_j)$$

minimum weighted c-balanced k-way cut density of G, where $0 < c \le 1/k$.

Remark:

$$\mu_k(G) = \min_{P\in\mathcal{P}_k} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \beta_{ij}(G/P),$$

where the weighted graph G/P is the k-quotient of G with respect to P.

Factor graph

The factor graph or k-quotient of G with respect to the k-partition P is denoted by G/P and it is defined as the weighted graph on k vertices with vertex- and edge-weights

$$\alpha_i(G/P) = \frac{\alpha_{V_i}}{\alpha_G} \quad (i = 1, \dots, k)$$

and

$$\beta_{ij}(G/P) = \frac{e_G(V_i, V_j)}{\alpha_{V_i}\alpha_{V_j}} \quad (i, j = 1, \dots, k),$$

respectively.

Relation to the ground state energies

Given the real symmetric $k \times k$ matrix $\mathbf{J}$ and the vector $\mathbf{h} \in \mathbb{R}^k$, the partitions $P \in \mathcal{P}_k$ also define a spin system on the weighted graph G. The so-called ground state energy of such a spin configuration is

$$\mathcal{E}_k(G, \mathbf{J}, \mathbf{h}) =$$
$$- \max_{P \in \mathcal{P}_k} \left(\sum_{i=1}^{k} \alpha_i(G/P) h_i + \sum_{i,j=1}^{k} \alpha_i(G/P)\alpha_j(G/P)\beta_{ij}(G/P) J_{ij} \right) .$$

Here $\mathbf{J}$ is the so-called coupling-constant matrix, where J_{ij} represents the strength of interaction between states i and j, and $\mathbf{h}$ is the magnetic field. They carry physical meaning. We shall use only special $\mathbf{J}$ and $\mathbf{h}$, especially $\mathbf{h} = \mathbf{0}$.

The microcanonical ground state energy of G given $\mathbf{a}$ and $\mathbf{J}$ ($\mathbf{h} = \mathbf{0}$) is

$$\mathcal{E}_k^{\mathbf{a}}(G, \mathbf{J}) = -\max_{P \in \mathcal{P}_k^{\mathbf{a}}} \sum_{i,j=1}^{k} \alpha_i(G/P)\alpha_j(G/P)\beta_{ij}(G/P)J_{ij}.$$

In a theorem Lovász et al. it is proved that the convergence of the weighted graph sequence (G_n) with no dominant vertex-weights is equivalent to the convergence of its microcanonical ground state energies for any k, $\mathbf{a}$, and $\mathbf{J}$.
Under the same conditions, the convergence of the above (G_n) implies the convergence of its ground state energies for any k, $\mathbf{J}$, and $\mathbf{h}$; further the convergence of the spectrum of (G_n).

Testability of the minimum multiway cut densities

$f_k(G)$ is testable, though $f_k(G_n) \to 0$ if there is no dominant node-weight. So, this is of not much use.

$f_k^{\mathbf{a}}(G)$ is testable for any $k \le |V(G)|$ and and distribution $\mathbf{a}$ over $\{1, \dots, k\}$.

$f_k^{c}(G)$ is testable for any $k \le |V(G)|$ and $c \le 1/k$.

The Newman–Girvan modularity is a special ground state energy, hence, its balanced versions are testable.

Testability of the weighted minimum multiway cut densities

μ_k is not testable:
We can show an example where $\mu_k(G_n) \to 0$, but randomizing a sufficiently large part of G_n, the weighted minimum k-way cut density of that part is constant.
The testability of μ_k^c can be proved in the same way as that of f_k^c.
The testability of $\mu_k^{\mathbf{a}}(G)$ also follows from the equivalent statements of testability.

Application of testability for fuzzy clustering

We proved that f_k^c is a testable weighted graph parameter. Now, we extend it to graphons.

$$\tilde{f}_k^c(W) := \inf_{S_1,\dots S_k} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \iint_{S_i \times S_j} W(x,y)\,dx\,dy,$$

where the infimum is taken over all the c-balanced Lebesgue-measurable partitions $(S_1,\dots,S_k)$ of [0,1]: $\sum_{i=1}^{k} \lambda(S_i) = 1$ and $\lambda(S_i) \geq c$, where λ denotes the Lebesgue-measure.

$\tilde{f}_k^c$ is the extension of f_k^c in the following sense: If (G_n) is a convergent weighted graph sequence with uniformly bounded edge-weights and no dominant vertex-weights, then denoting by W the limit graphon of the sequence, $f_k^c(G_n) - \tilde{f}_k^c(W)$ as $n \to \infty$.

The equivalent statements of testability use an essentially unique extension of a testable graph parameter to graphons. Therefore, the above $\tilde{f}_k^c$ is the desired extension of f_k^c and by the Equivalence Theorem: for a weighted graph sequence (G_n) with $\max_i \frac{\alpha_i(G_n)}{\alpha_{G_n}} \to 0$, the limit relation $\tilde{f}_k^c(W_{G_n}) - f_k^c(G_n) \to 0$ also holds as $n \to \infty$.

This gives rise to approximate the minimum c-balanced k-way cut density of a weighted graph on "many" vertices with no dominant vertex weights by the extended c-balanced k-way cut density of the stepfunction graphon assigned to the graph. In this way, the discrete optimization problem can be formulated as a quadratic programming task with linear equality and inequality constraints.

To this end, let us investigate a fixed weighted graph G on n vertices (n is large). As $f_k^c(G)$ is invariant under the scale of the vertices, we can suppose that $\alpha_G = \sum_{i=1}^n \alpha_i = 1$. As $\beta_{ij} \in [0,1]$, W_G is uniformly bounded by 1. Recall that $W_G(x,y) = \beta_{ij}$, if $x \in I_i$, $y \in I_j$, where $\lambda(I_j) = \alpha_j$ ($j = 1, \dots, n$) and $I_1, \dots, I_n$ are consecutive intervals of [0,1].
$\tilde{f}_k(W_G; S_1, \dots, S_k)$ is a continuous function over c-balanced k-partitions of [0,1] in the variables

$$\mathbf{x} = (x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, \dots, x_{k1}, \dots, x_{kn})^T \in \mathbb{R}^{nk}$$

where the coordinate indexed by ij is

$$x_{ij} = \lambda(S_i \cap I_j), \quad j = 1, \dots, n; \quad i = 1, \dots k.$$

$$\tilde{f}_k(W_G; S_1, \dots, S_k) = \frac{1}{2}\mathbf{x}^T(\mathbf{A} \otimes \mathbf{B})\mathbf{x},$$

where – denoting by $\mathbf{1}_{k\times k}$ and $\mathbf{I}_{k\times k}$ the $k \times k$ all 1's and the identity matrix, respectively – the eigenvalues of the $k \times k$ symmetric matrix $\mathbf{A} = \mathbf{1}_{k\times k} - \mathbf{I}_{k\times k}$ are the number $k-1$ and -1 with multiplicity $k-1$, while those of the $n \times n$ symmetric matrix $\mathbf{B} = (\beta_{ij})$ are $\lambda_1 \geq \cdots \geq \lambda_n$. Latter one being a Frobenius-type matrix, $\lambda_1 > 0$. The eigenvalues of the Kronecker-product $\mathbf{A} \otimes \mathbf{B}$ are the numbers $(k-1)\lambda_i$ $(i = 1, \dots, n)$ and $-\lambda_i$ with multiplicity $k-1$ $(i = 1, \dots, n)$. Therefore the above quadratic form is indefinite.

We have the following quadratic programming task:

$$\text{minimize} \qquad \tilde{f}_k(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T(\mathbf{A} \otimes \mathbf{B})\mathbf{x}$$

$$\text{subject to} \quad \mathbf{x} \geq 0; \quad \sum_{i=1}^{k} x_{ij} = \alpha_j \quad (j \in [n]); \quad \sum_{j=1}^{n} x_{ij} \geq c \quad (i \in [k]).$$

The feasible region is the closed convex polytope, and it is, in fact, in an $n(k-1)$-dimensional hyperplane of $\mathbb{R}^{nk}$. The gradient of the objective function $\nabla \tilde{f}_k(\mathbf{x}) = (\mathbf{A} \otimes \mathbf{B})\mathbf{x}$ cannot be $\mathbf{0}$ in the feasible region, provided the weight matrix $\mathbf{B}$, and hence $\mathbf{A} \otimes \mathbf{B}$ is not singular.

The arg-min of the quadratic programming task is one of the Kuhn–Tucker points (giving relative minima of the indefinite quadratic form over the feasible region), that can be found by numerical algorithms (by tracing back the problem to a linear programming task). In this way, for large n, we also get the solution of the following fuzzy clustering problem: let $x_{ij}/\lambda(S_i)$ denote the probability/proportion that vertex j belongs to cluster i. We find the optimum solution via quadratic programming. The index i giving the largest proportion can be regarded as the cluster membership of vertex j.
The problem can be solved with other equality or inequality constraints too.

Cut-norm of the Wigner-noise

Theorem

For any sequence $(G_{\mathbf{W}_n})$ of Wigner-graphs

$$\lim_{n\to\infty} \|W_{G_{\mathbf{W}_n}}\|_{\square} = 0 \qquad (n \to \infty) \quad \textit{almost surely.}$$

Corollary

Let $\mathbf{A}_n := \mathbf{B}_n + \mathbf{W}_n$ *and* $n_1, \dots, n_k \to \infty$ *in such a way that* $\lim_{n\to\infty} \frac{n_i}{n} = r_i$ $(i = 1, \dots, k)$, $n = \sum_{i=1}^{k} n_i$*; further, the uniform bound* K *of the entries of the "noise" matrix* $\mathbf{W}_n$ *is such that the entries of* $\mathbf{A}_n$ *are nonnegative. Under these conditions, the above Theorem implies that the "noisy" graph sequence* $(G_{\mathbf{A}_n}) \subset \mathcal{G}$ *converges almost surely in the* $\delta_\Box$ *metric. It is easy to see that the almost sure limit is the stepfunction* W_H*, where the factor graph* $H = G_{\mathbf{B}_n}/P$ *does not depend on* n*, as* P *is the* k*-partition of the vertices of* $G_{\mathbf{B}_n}$ *with resepect to the blow-up (with cluster sizes* $n_1, \dots, n_k$*). Actually, the vertex- and edge-weights of the weighted graph* H *are*

$$\alpha_i(H) = r_i \quad (i \in [k]), \qquad \beta_{ij}(H) = \frac{n_i n_j p_{ij}}{n_i n_j} = p_{ij} \quad (i, j \in [k]).$$

Generalized quasirandom graphs

H: model graph on k vertices with vertex-weights $r_1, \dots, r_k$ and edge-weights $p_{ij} = p_{ji}$ $(i, j = 1, \dots, k)$.
(G_n) is H-quasirandom if $G_n \to W_H$ as $n \to \infty$.
Lovász and Sós prove that the vertex set V of a generalized quasirandom graph G_n can be partitioned as $V_1, \dots, V_k$ in such a way that

- $\frac{|V_i|}{|V|} \to r_i$ $(i = 1, \dots, k)$
- the subgraph of G_n induced by V_i is a quasirandom graph with edge density p_{ii} $(i = 1, \dots, k)$
- the bipartite graphs between V_i and V_j are bipartite quasirandom with edge-density p_{ij} $(i \neq j)$

Generalized k-quasirandom properties

- The adjacency spectrum of G_n has k structural eigenvalues of order n and the others are $o(n)$; the k-variance of the vertex-representatives based on the eigenvectors corresponding to the structural eigenvalues is $\mathcal{O}(\frac{1}{n})$.
- There exists a $\delta \in (0,1)$ s.t. there are exactly $k-1$ structural eigenvalues of the normalized modularity spectrum greater than $\delta - o(1)$, and all the other eigenvalues are $o(1)$ in absolute value; the k-variance of the vertex-representatives based on the structural eigenvectors is $o(1)$.
- The vertices can be divided into clusters $V_1, \dots, V_k$ s.t. the V_i, V_j $(i \neq j)$ pairs are ε-volume regular, i.e., for all $X \subset V_i$, $Y \subset V_j$ satisfying $\texttt{Vol}(X) > \varepsilon \texttt{Vol}(V_i)$, $\texttt{Vol}(Y) > \varepsilon \texttt{Vol}(V_j)$:

$$|e(X,Y) - \frac{e(V_i,V_j)}{\texttt{Vol}(V_i)\texttt{Vol}(V_j)} \texttt{Vol}(X)\texttt{Vol}(Y)| \le \varepsilon \texttt{Vol}(V_i)\texttt{Vol}(V_j).$$

We found an exact relation between ε and the spectral gap in the normalized modularity spectrum.