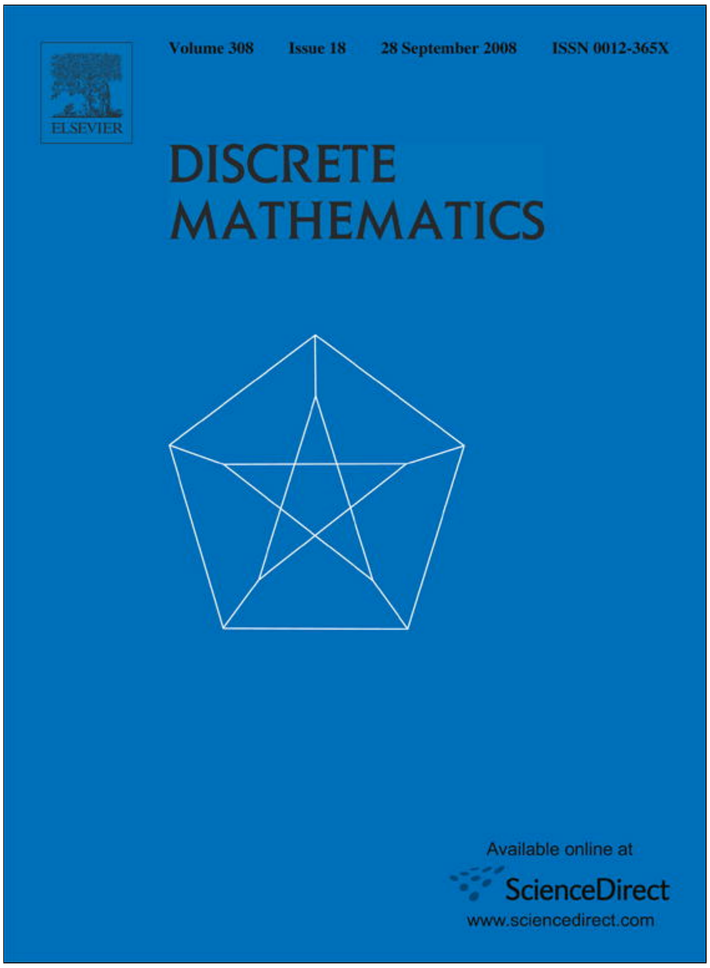
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Noisy random graphs and their Laplacians

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Abstract

Spectra and representations of some special weighted graphs are investigated with weight matrices consisting of homogeneous blocks. It is proved that a random perturbation of the weight matrix or that of the weighted Laplacian with a "Wigner-noise" will not have an effect on the order of the protruding eigenvalues and the representatives of the vertices will unveil the underlying block-structure.

Such random graphs adequately describe some biological and social networks, the vertices of which belong either to loosely connected strata or to clusters with homogeneous edge-densities between any two of them, like the structure guaranteed by the Regularity Lemma of Szemerédi.

MSC: 05C50; 15A42

Keywords: Spectra of weighted graphs; Wigner-noise; Weighted Laplacian; Graph representation

1. Introduction

Facing real-world graphs the number of vertices is sometimes so large that it is hard to discover their underlying structure. However, spectral techniques can tell us a lot about the graph. We say that our graph has a linear structure if its adjacency-, weight-, or Laplacian matrix can be well approximated by a lower rank matrix. This property can be characterized by the spectral decomposition of the matrix in question: it has some protruding eigenvalues and the Euclidean representatives of the vertices by means of the corresponding eigenvectors form well-separated clusters. In [6] we introduced the notion of Wigner-noise that is a generalization of a random matrix investigated by Wigner [11] and the bulk spectrum of which obeys the semi-circle law (if the size of the matrix tends to infinity). We recall the definition.

Definition 1.1. The $n \times n$ real matrix $\mathbf{W}$ is a symmetric Wigner-noise if its entries w_{ij}, $1 \leqslant i \leqslant j \leqslant n$, are independent random variables, $\mathbb{E}w_{ij} = 0$, Var $w_{ij} \leqslant \sigma^2$ with some $0 < \sigma < \infty$ and the w_{ij}'s are uniformly bounded (there is a constant $K > 0$ such that $|w_{ij}| \leqslant K$).

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On the basis of Füredi and Komlós [8], it was proved in [1] that for the maximum absolute value eigenvalue of $\mathbf{W}$

$$\max_{1\leqslant i\leqslant n} |\lambda_i(\mathbf{W})| \leqslant 2\sigma\sqrt{n} + \mathrm{O}(n^{1/3}\ \log n)$$

holds with probability tending to 1, as $n \to \infty$.

In the sequel, we put this noise on the following deterministic structures. The structures will be described by weight matrices assigned to special weighted graphs. Let $G_n = (V, \mathbf{B})$ be a weighted graph on the n element vertex set V, where the interconnections between the vertices are uniquely defined by the $n \times n$ symmetric weight matrix $\mathbf{B}$ (for its entries $0 \leqslant b_{ij} \leqslant 1$ are assumed and loops are allowed).

(i) The $n \times n$ symmetric weight matrix of G_n is defined in the following way. Let $k < n$ be a fixed integer, $\mu_i > 0$ and ν_i $(i = 1, \ldots, k)$ be real numbers, and the integers $n_1, \ldots, n_k > 0$ be such that $\sum_{i=1}^{k} n_i = n$. Let $\mathbf{B}$ be the Kronecker-sum of the matrices $\mathbf{B}_i$, $i = 1, \ldots, k$, where $\mathbf{B}_i$ is the $n_i \times n_i$ symmetric matrix with non-diagonal entries μ_i's and diagonal ones ν_i's (the non-diagonal blocks contain zeroes). Hence, G_n consists of k connected components, and within these components, each pair of edges is connected with an edge of the same weight. We put a Wigner-noise $\mathbf{W}$ of the corresponding size on the weight matrix $\mathbf{B}$. The spectral properties of the weight matrix $\mathbf{A} = \mathbf{B} + \mathbf{W}$ of the random graph $G'_n = (V, \mathbf{A})$ are investigated when the numbers k, K, σ, μ_i, and ν_i $(i = 1, \ldots, k)$ are kept fixed as $n_1, \ldots, n_k$ tend to infinity in such a way that $n_i/n \geqslant C$ $(i = 1, \ldots, k)$, with $n = \sum_{i=1}^{k} n_i$ and with some constant C $(0 < C < 1)$. In the sequel, we shall refer to this condition as *growth rate condition*. In [5] it was proved that with probability tending to 1 for the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $\mathbf{A}$ the following inequalities hold, as $n \to \infty$. There is an ordering of the k largest eigenvalues $\lambda_1, \ldots, \lambda_k$ such that

$$|\lambda_i - [(n_i - 1)\mu_i + \nu_i]| \leqslant 2\sigma\sqrt{n} + \mathrm{O}(n^{1/3}\ \log n), \quad i = 1, \ldots, k.$$

Among the other eigenvalues, for $i = 1, \ldots, k$ there are $n_i - 1$ λ_j's with

$$|\lambda_j - [\nu_i - \mu_i]| \leqslant 2\sigma\sqrt{n} + \mathrm{O}(n^{1/3}\ \log n).$$

It implies that the k largest eigenvalues of the random matrix $\mathbf{A}$ are of order $\Theta(n)$, and there must be a spectral gap between the k largest and the remaining eigenvalues with probability tending to 1, as $n \to \infty$. Asymptotic k-variate normality for the random vector $(\lambda_1, \ldots, \lambda_k)$ was also proved, and it was shown that the k-variance of the vertices of the perturbed graph G'_n—in the Euclidean representation defined by the corresponding eigenvectors—is $\mathrm{O}(1/n)$. The notion of k-variance and this kind of representation will be explained thoroughly in Section 2. For instance, such data structures occur, when the n observations come from k loosely connected strata $(k < n)$. For example, in [9] the importance of weak links between social strata is emphasized. In our model the weak links correspond to the entries of the Wigner-noise.

(ii) Let the $n \times n$ weight matrix $\mathbf{B}$ of G_n have a more general structure. In Section 2 it will be defined as a so-called blown up matrix of the $k \times k$ symmetric pattern matrix $\mathbf{P}$ (with the corresponding partition $V_1, \ldots, V_k$ of the vertex set $\{1, \ldots, n\}$): all the edge weights between vertices of the vertex clusters V_i and V_j are equal to p_{ij} $(i, j = 1, \ldots, k)$. In [6] it was proved that the $n \times n$ matrix $\mathbf{B}$ has k non-zero eigenvalues of order $\Theta(n)$ and the corresponding eigenvectors have equal coordinates for vertices belonging to the same cluster V_i (in other words, they have *piecewise constant structure*).

Now let G'_n be a random graph with weight matrix $\mathbf{A} = \mathbf{B} + \mathbf{W}$, where $\mathbf{W}$ is an $n \times n$ Wigner-noise. The integer k is kept fixed as $n_1, \ldots, n_k$ tend to infinity under the growth rate condition of (i). In Section 2 we shall prove that the weight matrix $\mathbf{A}$ still has k protruding eigenvalues of order $\Theta(n)$ and the representatives of the vertices of the noisy weighted graph $G'_n = (V, \mathbf{A})$ can be well classified into k clusters with probability tending to 1, as $n \to \infty$. It means that there are almost homogeneous edge-densities between any two of the clusters of the random graph G'_n. With an appropriate Wigner-noise our perturbed graph is a usual random graph with weights 1 or 0. For example, by adding a Wigner-noise we can reach that in the random graph $G'_n = (V, \mathbf{A})$ any pair of vertices of V_i and V_j is connected with probability p_{ij}. The Regularity Lemma of Szemerédi [10] guarantees a similar structure with some (perhaps large) k. Its relation to our problem is discussed in Section 4.

In Section 3 Laplacian spectra of graphs with weight matrices described in (i) and (ii) are investigated. Note that we can treat random perturbations only if the so-called weighted Laplacian of [4] is used.

2. Spectra of blown up weighted graphs

We cite the notion of a blown up matrix introduced in [6].

Definition 2.1. The $n \times n$ matrix $\mathbf{B}$ is a *blown up matrix* if there is a constant $k < n$, a $k \times k$ symmetric pattern matrix $\mathbf{P}$ with entries $0 \leqslant p_{ij} \leqslant 1$, and there are positive integers $n_1, \dots, n_k$, $\sum_{i=1}^{k} n_i = n$ such that $\mathbf{B}$ can be divided into k^2 blocks, the block (i, j) being an $n_i \times n_j$ matrix with entries all equal to p_{ij} $(1 \leqslant i, j \leqslant k)$.

As far as the spectrum of $\mathbf{B}$ is concerned, we repeat the statement and sketch a proof different from that of [6] so that it can be used to find the spectra of weighted Laplacian matrices in Section 3.

Proposition 2.2. *Under the growth rate condition all the non-zero eigenvalues of the $n \times n$ blown up matrix $\mathbf{B}$ are of order n in absolute value.*

Proof. As there are at most k linearly independent rows in $\mathbf{B}$, $r = \text{rank}\,(\mathbf{B}) \leqslant \text{k}$.

Let $\beta_1, \dots, \beta_r > 0$ be the non-zero eigenvalues of $\mathbf{B}$ and $\mathbf{u}_1, \dots, \mathbf{u}_r \in \mathbb{R}^n$ be the corresponding orthonormal eigenvectors. We drop the subscripts: let $\beta \neq 0$ be an eigenvalue with eigenvector of $\|\mathbf{u}\| = 1$. It is easy to see that $\mathbf{u}$ has piecewise constant structure: it has n_i coordinates equal to $u(i)$ $(i = 1, \dots, k)$. Then, with these coordinates the eigenvalue–eigenvector equation

$$\mathbf{Bu} = \beta \cdot \mathbf{u}$$

has the form

$$\sum_{j=1}^{k} n_j p_{ij} u(j) = \beta \cdot u(i) \quad (i = 1, \dots, k). \tag{1}$$

With the notations

$$\tilde{\mathbf{u}} = (u(1), \dots, u(k))^{\mathrm{T}}, \quad \mathbf{D} = \text{diag}\,(n_1, \dots, n_k), \tag{2}$$

Eq. (1) can be written in the form

$$\mathbf{PD}\tilde{\mathbf{u}} = \beta \cdot \tilde{\mathbf{u}}.$$

Further, introducing the transformation

$$\mathbf{v} = \mathbf{D}^{1/2}\tilde{\mathbf{u}}, \tag{3}$$

the equivalent equation

$$\mathbf{D}^{1/2}\mathbf{PD}^{1/2}\mathbf{v} = \beta \cdot \mathbf{v} \tag{4}$$

is obtained. It is easy to see that the transformation (3) results in a unit-norm vector. Furthermore, applying the transformation (3) for the $\tilde{\mathbf{u}}_i$ vectors obtained from the $\mathbf{u}_i$ $(i = 1, \dots, r)$, the orthogonality is also preserved.

Consequently, $\mathbf{v}_i = \mathbf{D}^{1/2}\tilde{\mathbf{u}}_i$ is an eigenvector corresponding to the eigenvalue β_i of the $k \times k$ matrix $\mathbf{D}^{1/2}\mathbf{PD}^{1/2}$ $(i = 1, \dots, r)$. With the shrinking

$$\widetilde{\mathbf{D}} = \frac{1}{n}\mathbf{D}. \tag{5}$$

Eq. (4) is also equivalent to

$$\widetilde{\mathbf{D}}^{1/2}\mathbf{P}\widetilde{\mathbf{D}}^{1/2}\mathbf{v} = \frac{\beta}{n} \cdot \mathbf{v},$$

that is the $k \times k$ matrix $\widetilde{\mathbf{D}}^{1/2}\mathbf{P}\widetilde{\mathbf{D}}^{1/2}$ has non-zero eigenvalues β_i/n with orthonormal eigenvectors $\mathbf{v}_i$ $(i = 1, \dots, r)$.

Now we want to establish relations between the eigenvalues of $\mathbf{P}$ and $\widetilde{\mathbf{D}}^{1/2}\mathbf{P}\widetilde{\mathbf{D}}^{1/2}$. Let $s_i(\mathbf{Q})$ denote the ith largest singular value of a matrix $\mathbf{Q}$. By the Courant–Fischer–Weyl minimax principle (cf. [3, p. 75])

$$s_i(\mathbf{Q}) = \max_{\dim H=i} \min_{\mathbf{x}\in H} \frac{\|\mathbf{Qx}\|}{\|\mathbf{x}\|}.$$

Since the singular values of a symmetric matrix are the absolute values of its real eigenvalues, and we are interested only in the first r eigenvalues, where $r = \operatorname{rank} \mathbf{B} = \operatorname{rank} \widetilde{\mathbf{D}}^{1/2}\mathbf{P}\widetilde{\mathbf{D}}^{1/2}$, it is sufficient to consider vectors $\mathbf{x}$, for which $\widetilde{\mathbf{D}}^{1/2}\mathbf{P}\widetilde{\mathbf{D}}^{1/2}\mathbf{x} \neq \mathbf{0}$ and apply the minimax principle with $i \in \{1, \dots, r\}$ and an arbitrary i-dimensional subspace $H \subset \mathbb{R}^k$:

$$\min_{\mathbf{x}\in H} \frac{\|\widetilde{\mathbf{D}}^{1/2}\mathbf{P}\widetilde{\mathbf{D}}^{1/2}\mathbf{x}\|}{\|\mathbf{x}\|} = \min_{\mathbf{x}\in H} \frac{\|\widetilde{\mathbf{D}}^{1/2}\mathbf{P}\widetilde{\mathbf{D}}^{1/2}\mathbf{x}\|}{\|\mathbf{P}\widetilde{\mathbf{D}}^{1/2}\mathbf{x}\|} \cdot \frac{\|\mathbf{P}\widetilde{\mathbf{D}}^{1/2}\mathbf{x}\|}{\|\widetilde{\mathbf{D}}^{1/2}\mathbf{x}\|} \cdot \frac{\|\widetilde{\mathbf{D}}^{1/2}\mathbf{x}\|}{\|\mathbf{x}\|}$$

$$\geqslant s_k(\widetilde{\mathbf{D}}^{1/2}) \cdot \min_{\mathbf{x}\in H} \frac{\|\mathbf{P}\widetilde{\mathbf{D}}^{1/2}\mathbf{x}\|}{\|\widetilde{\mathbf{D}}^{1/2}\mathbf{x}\|} \cdot s_k(\widetilde{\mathbf{D}}^{1/2}) \geqslant C \cdot \min_{\mathbf{x}\in H} \frac{\|\mathbf{P}\widetilde{\mathbf{D}}^{1/2}\mathbf{x}\|}{\|\widetilde{\mathbf{D}}^{1/2}\mathbf{x}\|},$$

with the constant C in the growth rate condition. Now taking the maximum for all possible i-dimensional subspace H we obtain that $|\lambda_i(\widetilde{\mathbf{D}}^{1/2}\mathbf{P}\widetilde{\mathbf{D}}^{1/2})| \geqslant C \cdot |\lambda_i(\mathbf{P})| > 0$. On the other hand,

$$|\lambda_i(\widetilde{\mathbf{D}}^{1/2}\mathbf{P}\widetilde{\mathbf{D}}^{1/2})| \leqslant \|\widetilde{\mathbf{D}}^{1/2}\mathbf{P}\widetilde{\mathbf{D}}^{1/2}\| \leqslant \|\widetilde{\mathbf{D}}^{1/2}\| \cdot \|\mathbf{P}\| \cdot \|\widetilde{\mathbf{D}}^{1/2}\| \leqslant \|\mathbf{P}\| \leqslant k.$$

These imply that $\lambda_i(\widetilde{\mathbf{D}}^{1/2}\mathbf{P}\widetilde{\mathbf{D}}^{1/2})$ is a non-zero constant, and because of $\lambda_i(\widetilde{\mathbf{D}}^{1/2}\mathbf{P}\widetilde{\mathbf{D}}^{1/2}) = \beta_i/n$ we obtain that $\beta_1, \dots, \beta_r = \Theta(n)$. □

In the following special case, in [6] we proved a little bit more:

Proposition 2.3. *Let the entries of the $k \times k$ pattern matrix be the following*: $p_{ii} = 0$ $(i = 1, \dots, k)$ *and* $p_{ij} = p_{ji} = p \in [0, 1]$ $(1 \leqslant i < j \leqslant k)$. *Let* $\mathbf{B}$ *be the blown up matrix of* $\mathbf{P}$ *with block sizes* $n_1 \leqslant n_2 \leqslant \dots \leqslant n_k$, $n := \sum_{i=1}^k n_i$. *Then* $\mathbf{B}$ *has exactly* $n - k$ *zero eigenvalues, the negative eigenvalues of* $\mathbf{B}$ *are in the interval* $[-pn_k, -pn_1]$, *while the positive ones in* $[p(n - n_k), p(n - n_1)]$.

We remark that in the case $p = 1$ our matrix $\mathbf{B}$ is the adjacency matrix of $K_{n_1,\dots,n_k}$, the complete k-partite graph on disjoint, edge-free vertex sets $V_1, \dots, V_k$ with $|V_i| = n_i$ $(i = 1, \dots, k)$.

Now let $\mathbf{B}$ be an $n \times n$ blown up matrix with non-zero eigenvalues $\beta_1, \dots, \beta_r$ and $\mathbf{W}$ be a symmetric Wigner-noise of the same size. Let G'_n be a random graph with weight matrix $\mathbf{A} = \mathbf{B} + \mathbf{W}$. We are looking for the spectral properties of $\mathbf{A}$, as k is kept fixed and $n_1, \dots, n_k$ tend to infinity under the usual growth rate condition. In [6], by an easy application of the Weyl's perturbation theorem, it was proved that under these conditions the weight matrix $\mathbf{A}$ still has r protruding eigenvalues: there are r eigenvalues $\lambda_1, \dots, \lambda_r$ of the noisy random matrix $\mathbf{A}$ such that

$$|\lambda_i - \beta_i| \leqslant 2\sigma\sqrt{n} + \mathrm{O}(n^{1/3} \log n), \quad i = 1, \dots, r$$

and for the other $n - r$ eigenvalues

$$|\lambda_j| \leqslant 2\sigma\sqrt{n} + \mathrm{O}(n^{1/3} \log n), \quad j = r + 1, \dots, n$$

hold with probability tending to 1, as $n \to \infty$.

Consequently, taking into account the order $\Theta(n)$ of the non-zero eigenvalues of $\mathbf{B}$, there is a spectral gap between the r largest absolute value and the other eigenvalues of $\mathbf{A}$, this is of order $\Delta - 2\varepsilon$, where

$$\varepsilon := 2\sigma\sqrt{n} + \mathrm{O}(n^{1/3} \log n) \quad \text{and} \quad \Delta := \min_{1\leqslant i\leqslant r} |\beta_i|. \tag{6}$$

In general, $r = \operatorname{rank} \mathbf{B} = k$, and the above statement guarantees the existence of k protruding eigenvalues of $\mathbf{A}$. On this basis, we can also estimate the distances between the corresponding eigenspaces of the matrices $\mathbf{B}$ and $\mathbf{A} = \mathbf{B} + \mathbf{W}$. Let us denote the unit norm eigenvectors belonging to the eigenvalues $\beta_1, \dots, \beta_k$ of $\mathbf{B}$ by $\mathbf{u}_1, \dots, \mathbf{u}_k$ and those belonging to the eigenvalues $\lambda_1, \dots, \lambda_k$ of $\mathbf{A}$ by $\mathbf{x}_1, \dots, \mathbf{x}_k$. Let $F := \operatorname{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_k\} \subset \mathbb{R}^n$ be k-dimensional subspace and let $\operatorname{dist}(\mathbf{x}, F)$ denote the Euclidean distance between the vector $\mathbf{x} \in \mathbb{R}^n$ and the subspace F.

Proposition 2.4. *With the above notation the following estimate holds with probability tending to* 1 ($n \to \infty$) *for the sum of the squared distances between* $\mathbf{x}_1, \dots, \mathbf{x}_k$ *and* F:

$$\sum_{i=1}^{k} \operatorname{dist}^2(\mathbf{x}_i, F) \leqslant k \frac{\varepsilon^2}{(\Delta - \varepsilon)^2} = \mathrm{O}\left(\frac{1}{n}\right). \tag{7}$$

The order of the estimate follows from the order of ε and Δ of (6).

This implies the well-clustering property of the representatives of the vertices of G'_n in the following representation. Let $\mathbf{X}$ be the $n \times k$ matrix containing the eigenvectors $\mathbf{x}_1, \dots, \mathbf{x}_k$ in its columns. Let the k-dimensional representatives of the vertices be the row vectors of $\mathbf{X}$: $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)} \in \mathbb{R}^k$. Let $S_k^2(\mathbf{X})$ denote the k-variance—introduced in [4]—of these representatives in the clustering $V_1, \dots, V_k$:

$$S_k^2(\mathbf{X}) = \sum_{i=1}^{k} \sum_{j \in V_i} \|\mathbf{x}^{(j)} - \bar{\mathbf{x}}^{(i)}\|^2 \quad \text{where } \bar{\mathbf{x}}^{(i)} = \frac{1}{n_i} \sum_{j \in V_i} \mathbf{x}^{(j)}.$$

With the above notation for the k-variance of the representation of the noisy weighted graph $G'_n = (V, \mathbf{A})$ the relation

$$S_k^2(\mathbf{X}) = \mathrm{O}\left(\frac{1}{n}\right)$$

holds with probability tending to 1, as $n \to \infty$ under the usual growth rate condition.

That is, by Theorem 3 of [5] it can easily be seen that $S_k^2(\mathbf{X})$ is equal to the left-hand side of (7), therefore it is also of order $\mathrm{O}(1/n)$.

Hence, the addition of any kind of a Wigner-noise to a weight matrix that has a blown up structure $\mathbf{B}$ will not change the order of the protruding eigenvalues of the noisy weight matrix, and the block structure of $\mathbf{B}$ can be concluded from the representatives of the vertices (where the representation is obtained by means of the corresponding eigenvectors).

With an appropriate Wigner-noise we can also reach that our matrix $\mathbf{B} + \mathbf{W}$ in its (i, j)th block contains 1's with probability p_{ij}, and 0's otherwise. That is, for indices $1 \leqslant i < j \leqslant k$ and $l \in V_i$, $m \in V_j$ let

$$w_{lm} := \begin{cases} 1 - p_{ij} & \text{with probability } \ p_{ij}, \\ -p_{ij} & \text{with probability } \ 1 - p_{ij} \end{cases}$$

be independent random variables, and for $i = 1, \dots, k$ and $l, m \in V_i$ $(l \leqslant m)$ let

$$w_{lm} := \begin{cases} 1 - p_{ii} & \text{with probability } \ p_{ii}, \\ -p_{ii} & \text{with probability } \ 1 - p_{ii} \end{cases}$$

be also independent, otherwise $\mathbf{W}$ is symmetric. This $\mathbf{W}$ satisfies the conditions of Definition 1.1 with entries of zero expectation and bounded variance

$$\sigma^2 = \max_{1 \leqslant i \leqslant j \leqslant k} p_{ij}(1 - p_{ij}) \leqslant \frac{1}{4}.$$

So, the noisy weighted graph $G'_n = (V, \mathbf{B} + \mathbf{W})$ becomes a usual random graph that has an edge between vertices of V_i and V_j with probability p_{ij}, $1 \leqslant i \leqslant j \leqslant k$. In particular, the noisy graph with underlying structure $\mathbf{B}$ of Proposition 2.3 has no edges within V_i $(i = 1, \dots, k)$, and it has an edge between vertices of V_i and V_j with the same probability $p = p_{ij}$ $(i \neq j)$. In this case the above statements guarantee the existence of k protruding eigenvalues of the incidence matrix of this random graph, while the corresponding eigenvectors give rise to a Euclidean representation of the vertices revealing the vertex clusters $V_1, \dots, V_k$.

We remark that by means of the sharp concentration results of Alon et al. [2] for the eigenvalues of large symmetric special random matrices, stronger results—almost sure convergence in our statements—could be proved. Wherever convergence with probability tending to 1 (as $n \to \infty$) is stated, one could write with probability 1 with sufficiently large n.

3. Laplacian spectra of blown up graphs

The Laplacian belonging to the symmetric weight matrix $\mathbf{A}$ of a weighted graph on n vertices is defined by

$$L(\mathbf{A}) = D(\mathbf{A}) - \mathbf{A}, \tag{8}$$

where $D(\mathbf{A}) = \operatorname{diag}(d_1, \ldots, d_n)$ with $d_i = \sum_{j=1}^{n} a_{ij}$, $i = 1, \ldots, n$, see [4]. The Laplacian is always singular, more precisely, the multiplicity of its zero eigenvalue is equal to the number of connected components of the underlying graph. Let

$$0 = \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$$

be the eigenvalues of $L(\mathbf{A})$ with corresponding unit norm eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. If the graph is connected then $\mathbf{v}_1 = \mathbf{1}/\sqrt{n}$, where $\mathbf{1}$ is the all 1 vector. In [4] it was also proved that $\sum_{i=1}^{k} \lambda_i$ gives the minimum of the quadratic function

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a_{ij} \|\mathbf{x}_i - \mathbf{x}_j\|^2 \tag{9}$$

over the k-dimensional representations of the vertices with $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^k$ such that $\sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i^{\mathrm{T}} = \mathbf{I}_k$, and the minimum is attained if the representatives $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are row vectors of the $n \times k$ matrix with column vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$. The objective function (9) tends to be small if representatives of vertices connected by large-weight edges are close to each other in Euclidean metric.

In model (i), $G_n = (V, \mathbf{B})$ consists of k connected components, therefore its Laplacian spectrum is the union of those of the components. Moreover, the Laplacian spectrum of a component is a simple linear transformation of its adjacency spectrum, that is with the notation of Section 1

$$L(\mathbf{B}_i) = [(n-1)\mu_i + \nu_i]\mathbf{I}_{n_i} - \mathbf{B}_i, \quad i = 1, \ldots, k.$$

The multiplicity of the zero eigenvalue of $L(\mathbf{B})$ is k and the corresponding eigenspace is spanned by eigenvectors $\mathbf{v}_i$ $(i = 1, \ldots, k)$ such that $\mathbf{v}_i$ has zero coordinates except in the ith block, where it has coordinates equal to $1/\sqrt{n_i}$. According to [4], after perturbing $\mathbf{B}$ (i.e., adding "few, small-weight" edges that connect different components), the Laplacian spectrum of the noisy graph G'_n will contain k "small" (i.e., almost zero) eigenvalues and there is a spectral gap between the k smallest and the other eigenvalues. The representation with the help of the eigenvectors corresponding to the k smallest eigenvalues will reveal the block-structure, but the same calculations with a Wigner-noise cannot be applied here, as the terms of the perturbation are not completely arbitrary because of the condition imposed on the Laplacian matrix (its row sums are zeroes): $L(\mathbf{B} + \mathbf{W}) = L(\mathbf{B}) + L(\mathbf{W})$, but $L(\mathbf{W})$ is not a Wigner-noise any more.

In the special case of model (ii)—if our underlying graph G_n is the complete k-partite graph $K_{n_1,\ldots,n_k}$ $(\sum_{i=1}^{k} n_i = n)$—we found that its Laplacian spectrum consists of one zero, the number n with multiplicity $k-1$, and the numbers $n - n_i$ with respective multiplicities $n_i - 1$ $(i = 1, \ldots, k)$. The reduction by 1 comes from the orthogonality of the eigenvectors belonging to positive eigenvalues to the all 1 vector. It was also shown that the maximum of the objective function (9) over $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^{k-1}$ with constraints $\sum_{i=1}^{k} \mathbf{x}_i \mathbf{x}_i^{\mathrm{T}} = \mathbf{I}_k$ and $\sum_{i=1}^{k} \mathbf{x}_i = \mathbf{0}$ is the sum of the $k-1$ largest eigenvalues, that is $(k-1)n$, and it is attained with the representation by means of any $k-1$ pairwise orthogonal, unit norm eigenvectors within the eigenspace belonging to this largest eigenvalue. As any vector in this eigenspace has equal coordinates within the blocks $V_1, \ldots, V_k$, the representatives form k different points in the $(k-1)$-dimensional Euclidean space. If $p_{ii} = 0$ and $0 < p_{ij} = p < 1$ for $(i > j)$, the eigenvalues of the Laplacian $L(\mathbf{B})$ of the blown up matrix $\mathbf{B}$ are p times the eigenvalues of the complete k-partite graph's Laplacian on the same set of vertices $V = (V_1, \ldots, V_k)$ with the same eigenvectors. A small perturbation will result in a Laplacian with $k-1$ large eigenvalues (of order n), and a $(k-1)$-dimensional representation with the corresponding eigenvectors will result in k well-separated clusters of the representatives. Now maximization is needed, as in this special case of model (ii), representatives of vertices connected by large-weight edges tend to be close to each other.

In the general model (ii) the Laplacian $L(\mathbf{B})$ of the blown up matrix $\mathbf{B}$ has one zero eigenvalue with corresponding eigenvector $\mathbf{1}$, and the positive eigenvalues are as follows. There are $k-1$ eigenvalues, say, $\lambda_1, \ldots, \lambda_{k-1}$ with eigenvectors having equal coordinates, say, $y_1, \ldots, y_k$ within the blocks $V_1, \ldots, V_k$ that satisfy the eigenvalue–eigenvector equation

$$\sum_{j\neq i} n_j p_{ij} y_i - \sum_{j\neq i} n_j p_{ij} y_j = \lambda y_i, \quad i = 1, \ldots, k.$$

It is also the eigenvalue–eigenvector equation of the $k \times k$ formal Laplacian $L(\mathbf{D}^{1/2}\mathbf{P}\mathbf{D}^{1/2}) = L(n\tilde{\mathbf{D}}^{1/2}\mathbf{P}\tilde{\mathbf{D}}^{1/2})$, where $\mathbf{D}$ and $\tilde{\mathbf{D}}$ are defined in (2) and (5). Due to this coincidence, latter ones also have non-negative real eigenvalues: one zero, and $k-1$ positive numbers in the generic case. Let us denote the positive eigenvalues of $L(\tilde{\mathbf{D}}^{1/2}\mathbf{P}\tilde{\mathbf{D}}^{1/2})$ by $\alpha_1, \ldots, \alpha_{k-1}$. Therefore

$$\lambda_i = n\alpha_i = \Theta(n), \quad i = 1, \ldots, k-1,$$

and the corresponding eigenvectors are the blown up vectors of the eigenvectors of form $(y_1, \ldots, y_k)^{\mathrm{T}}$ satisfying $\sum_{i=1}^{k} n_i y_i = 0$ (due to the orthogonality to the vector $\mathbf{1} \in \mathbb{R}^n$).

The other positive eigenvalues of $L(\mathbf{B})$ are the numbers μ_i with multiplicity $n_i - 1$:

$$\mu_i = \sum_{j\neq i} n_j p_{ij} = n \sum_{j\neq i} \frac{n_j}{n} p_{ij} = \Theta(n), \quad i = 1, \ldots, k,$$

since the summation is only over $k-1$ indices and we use the growth rate condition. It can be easily seen that an eigenvector $\mathbf{v}_i$ belonging to μ_i with coordinates v_{ij} $(j = 1, \ldots, n)$ is such that $v_{ij} = 0$, if $j \notin V_i$ and $\sum_{j\in V_i} v_{ij} = 0$.

The non-zero eigenvalues are all of order $\Theta(n)$, therefore the eigenvectors belonging to the non-zero eigenvalues above are capable to reveal the underlying block-structure of the blown up graph, as they have either piecewise constant structure or zero coordinates except one cluster. However, Wigner-type perturbations cannot be treated in this case either. In [4] the notion of weighted Laplacian belonging to the $n \times n$ weight matrix $\mathbf{A} = (a_{ij})$ was also introduced that assigns the generalized degree $d_i = \sum_{j=1}^{n} a_{ij}$ to vertex i as weight. Sometimes this kind of a Laplacian matrix is more suitable for classification purposes and, as it is emphasized in this paper, for perturbation with Wigner-noise too.

Given a weighted graph on n vertices with weight matrix $\mathbf{A}$, its weighted Laplacian is defined as

$$L'(\mathbf{A}) = \mathbf{I}_n - \mathbf{D}(\mathbf{A})^{-1/2} \cdot \mathbf{A} \cdot \mathbf{D}(\mathbf{A})^{-1/2}.$$

In [4] the properties of $L'(\mathbf{A})$ were thoroughly discussed: its eigenvalues are in the [0,2] interval and the multiplicity of zero as an eigenvalue is also equal to the number of connected components of the underlying graph. It was also proved that the sum of the k smallest eigenvalues of $L'(\mathbf{A})$ gives the minimum of the quadratic function defined in (9) under the constraint $\sum_{i=1}^{n} d_i \mathbf{x}_i \mathbf{x}_i^{\mathrm{T}} = \mathbf{I}_k$.

Now let $G_n = (V, \mathbf{B})$ be a weighted graph with $n \times n$ blown up weight matrix $\mathbf{B}$ of the $k \times k$ symmetric pattern matrix $\mathbf{P}$. We want to characterize the spectrum of $L'(\mathbf{B})$. Without loss of generality, suppose that rank $(\mathbf{B})$ = rank $(\mathbf{P})$ = k, and $\mathbf{P}$ has no identically zero rows (it means that the underlying graph has no isolated vertices).

Proposition 3.1. *Under the growth rate condition there exists a constant $\delta \in (0, 1)$, independent of n, such that there are k eigenvalues of $L'(\mathbf{B})$ that are not equal to* 1 *and they are located in the union of intervals* $[0, 1-\delta]$ *and* $[1+\delta, 2]$.

Proof. It is easy to see that $\mathbf{D}(\mathbf{B})^{-1/2} \cdot \mathbf{B} \cdot \mathbf{D}(\mathbf{B})^{-1/2}$ is also a blown up matrix of the $k \times k$ symmetric pattern matrix $\tilde{\mathbf{P}}$ with entries

$$\tilde{p}_{ij} = \frac{p_{ij}}{\sqrt{(\sum_{l=1}^{k} p_{il} n_l)(\sum_{m=1}^{k} p_{mj} n_m)}}.$$

Following the considerations of the proof of Proposition 2.2, the blown up matrix $\mathbf{D}(\mathbf{B})^{-1/2} \cdot \mathbf{B} \cdot \mathbf{D}(\mathbf{B})^{-1/2}$ has exactly k non-zero eigenvalues that are the eigenvalues of the $k \times k$ symmetric matrix $\mathbf{P}' = \mathbf{D}^{1/2}\tilde{\mathbf{P}}\mathbf{D}^{1/2}$ with entries

$$p'_{ij} = \frac{p_{ij}\sqrt{n_i}\sqrt{n_j}}{\sqrt{(\sum_{l=1}^{k} p_{il} n_l)(\sum_{m=1}^{k} p_{mj} n_m)}} = \frac{p_{ij}}{\sqrt{(\sum_{l=1}^{k} p_{il} \frac{n_l}{n_j})(\sum_{m=1}^{k} p_{mj} \frac{n_m}{n_i})}}.$$

As $\mathbf{P}$ has no identically zero rows, the matrix $\mathbf{P}'$ varies on a compact set of $k \times k$ matrices determined, due to the growth rate condition, by the inequalities

$$C \leqslant \frac{n_i}{n_j} \leqslant \frac{1}{C} \quad (i, j = 1, \ldots, k).$$

The range of the non-zero eigenvalues depends continuously on the matrix that does not depend on n. Therefore, the minimum non-zero eigenvalue does not depend on n. The eigenvalues of $\mathbf{D}(\mathbf{B})^{-1/2} \cdot \mathbf{B} \cdot \mathbf{D}(\mathbf{B})^{-1/2}$ being in the $[-1, 1]$ interval, this finishes the proof. (If our matrix is subtracted from the identity matrix, the zero eigenvalue is transformed into 1, while the others are transformed into the right and left neighborhood of 0 and 2, respectively.) □

Proposition 3.1 states that the non-one eigenvalues of $L'(\mathbf{B})$ are strictly separated from 1. We claim that this property is inherited to the weighted Laplacian $L'(\mathbf{A})$ of the noisy graph $G'_n = (V, \mathbf{A})$, where $\mathbf{A} = \mathbf{B} + \mathbf{W}$ with a Wigner-noise of appropriate size. In fact, the following statement can be formulated.

Proposition 3.2. *There exists a positive number $\delta \in (0, 1)$, independent of n, such that for every $0 < \tau < \frac{1}{2}$ the following statement holds with probability tending to 1, as $n \to \infty$ under the growth rate condition: there are exactly k eigenvalues of $L'(\mathbf{A})$ that are located in the union of intervals $[-n^{-\tau}, 1 - \delta + n^{-\tau}]$ and $[1 + \delta - n^{-\tau}, 2 + n^{-\tau}]$, while all the others are in the interval $(1 - n^{-\tau}, 1 + n^{-\tau})$.*

This statement, as well as the following results are not proved here, as they are topics of a paper—dealing with more general types of random matrices—in preparation.

Let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ be unit-norm, pairwise orthogonal eigenvectors belonging to the non-one eigenvalues of $L'(\mathbf{B})$. The n-dimensional vectors obtained by the transformations

$$\mathbf{u}'_i := \mathbf{D}(\mathbf{B})^{-1/2}\mathbf{u}_i \quad (i = 1, \ldots, k)$$

have the following optimality property: the $n \times k$ matrix formed by them, as column vectors, contains the k-dimensional representatives $\mathbf{x}_1, \ldots, \mathbf{x}_n$ of vertices that minimize (9) under the constraint $\sum_{i=1}^n d_i \mathbf{x}_i \mathbf{x}_i^{\mathrm{T}} = \mathbf{I}_k$.

Let $0 < \tau < \frac{1}{2}$ be arbitrary and $\epsilon := n^{-\tau}$. Let us also denote the unit-norm, pairwise orthogonal eigenvectors corresponding to the k eigenvalues of $L'(\mathbf{A})$ separated from 1 by $\mathbf{y}_1, \ldots, \mathbf{y}_k \in \mathbb{R}^n$ (their existence is guaranteed by Proposition 3.2). Further set

$$F := \operatorname{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}.$$

Proposition 3.3. *With the above notation, the following estimate holds with probability tending to 1 ($n \to \infty$ under the growth rate condition) for the distance between $\mathbf{y}_i$ and F:*

$$\operatorname{dist}(\mathbf{y}_i, F) \leqslant \frac{\epsilon}{(\delta - \epsilon)} = \frac{1}{\left(\frac{\delta}{\epsilon} - 1\right)} \quad (i = 1, \ldots, k). \tag{10}$$

Observe that the statement is similar to that of Proposition 2.4 with δ instead of Δ and ϵ instead of ε. The right-hand side of (10) is of order $n^{-\tau}$ that tends to zero, as $n \to \infty$.

Proposition 3.3 implies the well-clustering property of the vertex representatives by means of the transformed eigenvectors

$$\mathbf{y}'_i := \mathbf{D}(\mathbf{A})^{-1/2}\mathbf{y}_i \quad (i = 1, \ldots, k).$$

Let $\mathbf{Y}'$ be the $n \times k$ matrix containing the vectors $\mathbf{y}'_1, \ldots, \mathbf{y}'_k$ in its columns. Let the k-dimensional representatives of the vertices be the row vectors of $\mathbf{Y}'$: $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(n)} \in \mathbb{R}^k$. With respect to the vertex weights $d_1, \ldots, d_n$ the k-variance of these representatives is defined by

$$S_k^2(\mathbf{Y}') = \sum_{i=1}^{k} \sum_{j \in V_i} d_j \|\mathbf{y}^{(j)} - \bar{\mathbf{y}}^{(i)}\|^2 \quad \text{where } \bar{\mathbf{y}}^{(i)} = \sum_{j \in V_i} d_j \mathbf{y}^{(j)}$$

and $V_1, \ldots, V_k$ is the partition of the vertices with respect to the blow up.

Proposition 3.4. *With the above notation,*

$$S_k^2(\mathbf{Y}') \leqslant \frac{r}{\left(\frac{\delta}{\epsilon} - 1\right)^2}$$

holds with probability tending to 1, *as* $n \to \infty$ *under the growth rate condition.*

Proof. An easy calculation shows that

$$S_k^2(\mathbf{Y}') = \sum_{i=1}^{k} \operatorname{dist}^2(\mathbf{y}_i, F)$$

and hence the result of Proposition 3.3 can be used. □

4. Conclusions and further remarks

Through the models introduced in Section 1 we have shown that if the adjacency matrix of our underlying graph G_n has some protruding eigenvalues (of order n in absolute value), then a Wigner-noise cannot disturb essentially this structure: with probability tending to 1 (as $n \to \infty$), the adjacency matrix of the noisy graph G'_n will have the same number of protruding eigenvalues with corresponding eigenvectors revealing the structure of the graph. A representation with them makes it possible to find the clusters, if our underlying graph consists of k loosely, strongly, or homogeneously connected components.

The Laplacians also reflect the underlying structure, but it depends on the model that eigenvectors belonging to the small, large or medium eigenvalues are to be chosen for the representation. In addition, the application of perturbation results is not straightforward for the Laplacian, but it can be done with the help of the weighted Laplacian. As we showed in Section 3, the noisy weighted graph (of the blown up structure burdened with Wigner-noise) also has a well-classifiable representation by means of its weighted Laplacian's spectral decomposition.

For any graph on n vertices the Regularity Lemma of Szemerédi guarantees the existence of a partition $V_0, V_1, \ldots, V_k$ of the vertices (here V_0 is a "small" exceptional set) such that most of the V_i, V_j pairs ($1 \leqslant i < j \leqslant k$) are ε-regular with $\varepsilon > 0$ fixed in advance. (A pair V_i, V_j ($i \neq j$) is ε-regular, if for any $A \subset V_i$, $B \subset V_j$ with $|A| > \varepsilon |V_i|$, $|B| > \varepsilon |V_j|$ the relation $|\operatorname{dens}(A, B) - \operatorname{dens}(V_i, V_j)| < \varepsilon$ holds, where $\operatorname{dens}(A, B)$ denotes the edge-density between the disjoint vertex-sets A and B. In fact, denoting by $\operatorname{cut}(A, B)$ the cut-set between A and B, $\operatorname{dens}(A, B) = |\operatorname{cut}(A, B)|/|A| \cdot |B|$.) In some sense, ε-regularity means that the edge-densities between the V_i, V_j pairs are homogeneous. If our random graph has a weight matrix $\mathbf{A} + \mathbf{W}$ (a blown up matrix plus a special Wigner-noise discussed at the end of Section 2), then $|\operatorname{cut}(V_i, V_j)|$ is the sum of $|V_i| \cdot |V_j|$ independent, identically distributed Bernoulli variables with parameter p_{ij} ($1 \leqslant i, j \leqslant k$), where p_{ij}'s are entries of the pattern matrix $\mathbf{P}$. Hence $|\operatorname{cut}(A, B)|$ is a binomially distributed random variable with expectation $|A| \cdot |B| \cdot p_{ij}$ and variance $|A| \cdot |B| \cdot p_{ij}(1 - p_{ij})$. We shall need the following lemma.

Lemma 4.1 (*Chernoff inequality for large deviations*). *Let* $X_1, \ldots, X_n$ *be independent random variables*, $|X_i| \leqslant K$, $X := \sum_{j=1}^{n} X_j$. *Then for any* $a > 0$:

$$\mathbb{P}(|X - \mathbb{E}(X)| > a) \leqslant \mathrm{e}^{-a^2/2(\operatorname{Var}(X) + Ka/3)}.$$

Therefore, with a fixed $\varepsilon > 0$, $K = 1$ and with $A \subset V_i$, $B \subset V_j$, $|A| > \varepsilon|V_i|$, $|B| > \varepsilon|V_j|$ we have that

$$\begin{aligned}\mathbb{P}(|\text{dens}\,(A, B) - p_{ij}| > \varepsilon) &= \mathbb{P}(|\text{cut}\,(A, B) - |A| \cdot |B| \cdot p_{ij}| > \varepsilon \cdot |A| \cdot |B|) \\ &\leqslant \mathrm{e}^{-\varepsilon^2|A|^2|B|^2/2[|A||B|p_{ij}(1-p_{ij})+\varepsilon|A||B|/3]} \\ &= \mathrm{e}^{-\varepsilon^2|A||B|/2[p_{ij}(1-p_{ij})+\varepsilon/3]} \\ &\leqslant \mathrm{e}^{-\varepsilon^4|V_i||V_j|/2[p_{ij}(1-p_{ij})+\varepsilon/3]}\end{aligned}$$

that tends to 0, as $|V_i| = n_i \to \infty$ and $|V_j| = n_j \to \infty$. Hence, any pair V_i, V_j is ε-regular with probability tending to 1, if $n_1, \ldots, n_k \to \infty$ under the growth rate condition.

In summary, if the adjacency matrix of the noisy random graph has k protruding eigenvalues (greater than $\sqrt{n}$ in order of magnitude) and the representatives of vertices (by means of the corresponding eigenvectors) form k well-separated clusters, we get a construction for the clusters of the Szemerédi's Regularity Lemma themselves (see [6] for further details of this algorithm). A Wigner-noise will not change these clusters. So, our graph is a so-called generalized random graph of [10]. An other kind of construction for the clusters of the lemma is discussed in [7].

In case of large real-life graphs one often looks for a blown up structure behind the edge-weights. For example, some communication networks and metabolic networks of cells on a large number of vertices may have homogeneous edge-densities between a fixed number of components, that is exposed to random noise, like weak links between social strata or mutations in cells.

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