POSITIVE FORMS ON BANACH SPACES

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Abstract

The first representation theorem establishes a correspondence between positive, self-adjoint operators and closed, positive forms on Hilbert spaces. The aim of this paper is to show that some of the results remain true if the underlying space is a reflexive Banach space. In particular, the construction of the Friedrichs extension and the form sum of positive operators can be carried over to this case.

1 Introduction

Let X denote a reflexive complex Banach space, and X^* its conjugate dual space (i.e. the space of all continuous, conjugate linear functionals over X). We will use the notation $(v, x) := v(x)$ for $v \in X^*$, $x \in X$, and $(x, v) := \overline{v(x)}$. Let A be a densely defined linear operator from X to X^*. Notice that in this context it makes sense to speak about positivity and self-adjointness of A. Indeed, A defines a sesquilinear form on $\operatorname{Dom} A \times \operatorname{Dom} A$ via $t_A(x, y) = (Ax)(y) = (Ax, y)$ and A is called positive if t_A is positive, i.e. if $(Ax, x) \geq 0$ for all $x \in \operatorname{Dom} A$. Also, the adjoint A^* of A is defined (because A is densely defined) and is a mapping from X^{**} to X^*, i.e. from X to X^*. Thus, A is called self-adjoint if $A = A^*$. Similarly, the operator A is called symmetric if the form t_A is symmetric.

In Section 2 we deal with closed, positive forms and associated operators, and we establish a generalized version of the first representation theorem. In Section 3 we apply the representation theorem in two situations: first we construct the Friedrichs extension of a positive, symmetric operartor, then we define the form sum of two positive, self-adjoint operators. With the help of a factorization lemma we give a definition of a partial order of positive, self-adjoint operators. The Friedrichs extension turns out to be the largest of all self-adjoint extensions with respect to this partial order. The factorization lemma also enables us to prove a certain commutation property of the form sum of positive self-adjoint operators. In the last section we give two applications of the results, one in the theory of partial differential equations and one in probability theory. Some results of this paper have already appeared in a not widely known paper [2].

2 Representation theorem

Let $D \subseteq X$ be a dense subspace, and let $t : D \times D \to \mathbb{C}$ be a sesquilinear form on D (where t is linear in the first variable and conjugate linear in the second). Assume that t is positive with positive lower bound, i.e. $t(x,x) \geq \gamma \|x\|^2,\ \gamma > 0$. Assume also that t is "closed" in the sense that $(D, t(\cdot,\cdot)) =: H$ is a Hilbert space (i.e. it is complete). In this case, the injection $i : H \to X$ is continuous so H can be regarded as a subspace of X. For brevity we will use the notation $[\cdot,\cdot]$ for $t(\cdot,\cdot)$. An operator A from X to X^* can be associated to the form t in a natural way: let $x \in D$ and take the functional $[x,y],\ y \in D$; if this functional is continuous in the norm of X then there is an element z in X^* for which $[x,y] = z(y) =: (z,y)$, in this case, let $Ax := z$.

Theorem 1 *With notations as above the operator $A : X \to X^*$ is a positive, self-adjoint operator.*

Proof. Let $v \in X^*$ be an arbitrary element. Now, (v,x) $x \in D$ is a continuous, conjugate linear functional on H. Indeed,

$$|(v,x)| \leq \|v\|\ \|x\| \leq \frac{1}{\sqrt{\gamma}}[x]\ \|v\| = K[x],$$

where $[x]$ denotes the norm of H, i.e. $[x] = [x,x]^{1/2}$. Thus, by the theorem of Riesz we have an element $f \in H$ such that $(v,x) = [f,x]$. Define an operator B from X^* to X by $Bv := f$. Then B is defined everywhere on X^*, and B is positive and bounded with $\|B\| \leq \frac{1}{\gamma}$. Indeed, $(z,Bz) = [Bz,Bz] = [Bz]^2 \geq 0$, and

$$\|Bz\|^2 \leq \frac{1}{\gamma}[Bz]^2 = \frac{1}{\gamma}(Bz,z) \leq \frac{1}{\gamma}\|Bz\|\ \|z\|.$$

Hence, B is a bounded, positive, self-adjoint operator. Furthermore, B is injective. To see this, suppose that $Bz = 0$. Then $0 = [Bz,g] = (z,g)$ for every $g \in H$, and H is dense in X therefore $z = 0$. This means that the inverse B^{-1} exists and is a linear mapping from X to X^*. We will show that $A = B^{-1}$. Let $x \in \mathrm{Dom} A$, then $[x,y] = (t,y)$ for some $t \in X^*$ and $Ax = t$. Also, $(t,y) = [Bt,y]$ so $Bt = x$, and hence $A \subseteq B^{-1}$. Conversely, if $x \in \mathrm{Dom}\, B^{-1}$ then $x = Bz$ for some $z \in X^*$ and $[x,y] = [Bz,y] = (z,y)$ is continuous in y therefore $x \in \mathrm{Dom}\, A$ and $Ax = z = B^{-1}z$, which proves that $B^{-1} \subseteq A$. To complete the proof we have the following lemma, which is well known in Hilbert spaces. ∎

Lemma 1 *If $B : X^* \to X$ is a bounded, injective, self-adjoint operator then $A := B^{-1}$ is also a self-adjoint operator from X to X^*.*

Proof. First we show that $\mathrm{Ran}\, B$ is dense in X. Indeed, if for some $v \in X^*$ we have $(Bz,v) = 0$ for every $z \in X^*$, then $(Bz,v) = (z,Bv) = 0$ so $Bv = 0$ and $v = 0$. Hence A is densely defined. Also, A is symmetric, because if $x \in \mathrm{Dom}\, A$ then $x = Bz$ for some $z \in X^*$ and $(Ax,x) = (z,Bz) \in \mathbb{R}$. Thus $A \subseteq A^*$. To see the reverse inclusion, let $y \in \operatorname{Dom} A^*$ and let $x = Bz$ run through the elements of Dom A. Then $(Ax, y) = (z, y)$ and also

$$(Ax, y) = (x, A^*y) = (Bz, A^*y) = (z, BA^*y)$$

which means that $y = BA^*y$, so $y \in \operatorname{Dom} A$. ∎

Remark. The previous arguments can be carried out whenever (X, Y) is a dual pair of locally convex, topological linear spaces. In this case, we replace the condition of the lower bound by the natural assumption that the injection i introduced above is continuous.

3 The Friedrichs extension and the form sum

In this section we apply the representation theorem in two situations. First we construct the Friedrichs extension of a densely defined positive operator.

Theorem 2 *Let $a : X \to X^*$ be a positive, densely defined operator with positive lower bound, $(ax, x) \geq \gamma\|x\|^2,\ \gamma > 0$ for every $x \in \operatorname{Dom} a$. Then a admits a positve self-adjoint extension with the same lower bound.*

Proof. The form $t_a(x, y) := (ax, y)$ defines a pre-Hilbert space on Dom$a$. Denote the completion of this space by H, and the arising inner product by $[\cdot, \cdot]$. The injection $i : \operatorname{Dom} a \to X$ extends by continuity to H and the extension will be denoted by I_a. We prove that I_a is injective. Notice first that $[t, y] = (at, I_a y)$ for all $t \in \operatorname{Dom} a$, $y \in H$. Indeed, take a sequence $y_n \in \operatorname{Dom} a$, $y_n \to y$ in H (which implies convergence in X as well), then

$$(at, I_a y) = \lim(at, I_a y_n) = \lim[t, y_n] = [t, y].$$

Now assume that $I_a y = 0$. Then

$$[y]^2 = \lim[y_n, y] = \lim(ay_n, I_a y) = 0$$

therefore $y = 0$ which means that I_a is injective. Thus H can be regarded as a subspace of X and Theorem 1 can be applied. It is clear that the arising self-adjoint operator A_F is an extension of a and we also see from the proof of Theorem 1 that $(A_F x, x) \geq \gamma\|x\|^2$ for all $x \in \operatorname{Dom} A$. This operator will be called the Friedrichs extension of a. ∎

Next we examine a characterizing property of the Friedrichs extension. In the Hilbert space setting the Friedrichs extension distinguishes itself by being the largest, self-adjoint extension with respect to the usual partial ordering of positive operators. The definition of this ordering is that for positive operators A and B we have $A \geq B$ if $\operatorname{Dom} A^{\frac{1}{2}} \subseteq \operatorname{Dom} B^{\frac{1}{2}}$ and $(A^{\frac{1}{2}}x, A^{\frac{1}{2}}) \geq (B^{\frac{1}{2}}x, B^{\frac{1}{2}})$ for all $x \in \operatorname{Dom} A^{\frac{1}{2}}$. We will now examine how this definition can be carried over to our situation. The following factorization lemma is well known in Hilbert spaces (see e.g. [9], [10]). For bounded positive self-adjoint operators from X to X^* this lemma was also proved in [11]. and it plays a key role in the characterization of covariance operators of Banach space valued random variables.

Lemma 2 *Let A be a positive self-adjoint operator from X to X^* (that is $(Ax, x) \geq 0$ for all $x \in \operatorname{Dom} A$). Then there exists an auxiliary Hilbert space H and an operator $J : H \to X^*$ such that $A = JJ^*$.*

Proof. Define an inner product on $\operatorname{Ran} A$ by $[Ax, Ay] := (Ax, y)$. It is well defined because if $Ax_1 = Ax_2$ and $Ay_1 = Ay_2$ then

$$(Ax_1, y_1) = (Ax_2, y_1) = (x_2, Ay_1) = (x_2, Ay_2) = (Ax_2, y_2).$$

Furthermore it is positive definite, because if $[Ax, Ax] = (Ax, x) = 0$ then by the Cauchy inequality we have

$$|(Ax, y)|^2 \leq (Ax, x)(Ay, y) = 0$$

for all $y \in \operatorname{Dom} A$ which implies that $Ax = 0$. Thus $(\operatorname{Ran} A, [\cdot, \cdot])$ is a pre-Hilbert space. Denote the completion of this space by H_A. Define the operator $J : H_A \to X^*$ by $\operatorname{Dom} J = \operatorname{Ran} A$ and $J(Ax) := Ax$ for all $Ax \in \operatorname{Ran} A$. Then, by definition $\operatorname{Dom} J^* = \{y \in X : |(Ax, y)|^2 \leq M_y(Ax, x) \text{ for all } x \in \operatorname{Dom} A\}$, in particular $\operatorname{Dom} A \subseteq \operatorname{Dom} J^*$ and $J^* y = Ay$ for all $y \in \operatorname{Dom} A$. Thus JJ^* is an extension of A and JJ^* is symmetric. It is also clear that a self-adjoint operator is maximal symmetric just as in the context of Hilbert spaces. This means that $A = JJ^*$. ∎

Remark. Notice that in the context of Hilbert spaces $\operatorname{Dom} A^{\frac{1}{2}} = \operatorname{Dom} J^*$ and $(A^{\frac{1}{2}} x, A^{\frac{1}{2}} x) = [J^* x, J^* x]$.

Now we are in position to give a definition of ordering without the use of square roots.

Definition 1 *For positive, self-adjoint operators A and B we say that $A \geq B$ if and only if $\operatorname{Dom} J_A^* \subseteq \operatorname{Dom} J_B^*$ and $[J_A^* y, J_A^* y]_A \geq [J_B^* y, J_B^* y]_B$ for all $y \in \operatorname{Dom} J_A^*$.*

In order to understand this definition better it would be desirable to give a characterization of $\operatorname{Dom} J_A^*$ and $[J_A^* y, J_A^* y]_A$ in terms of A only.

Lemma 3 *With notations as above we have*

$$\operatorname{Dom} J_A^* = \left\{ y \in X : \sup_{x \in \operatorname{Dom} A, (Ax, x) \leq 1} |(Ax, y)|^2 < \infty \right\}$$

and

$$[J_A^* y, J_A^* y]_A = \sup_{x \in \operatorname{Dom} A, (Ax, x) \leq 1} |(Ax, y)|^2$$

Proof. The characterization of $\operatorname{Dom} J^*$ is clear from

$$\operatorname{Dom} J^* = \{y \in X : |(Ax, y)|^2 \leq M_y(Ax, x) \text{ for all } x \in \operatorname{Dom} A\}.$$

To see the other equality notice that $\operatorname{Ran} A$ is dense in H_A, therefore we have

$$[J_A^* y, J_A^* y]_A = \sup_{(Ax, x) \leq 1} |[J^* y, Ax]|_A^2 = \sup_{(Ax, x) \leq 1} |(y, Ax)|^2$$

∎

Remark. Notice that this new definition of partial ordering coincides with the usual one when X is a Hilbert space. If we allow the value of the indicated supremum to be $+\infty$, then we can say in short that $A \geq B$ if and only if

$$\sup_{x \in \mathrm{Dom}\, A, (Ax,x) \leq 1} |(Ax, y)|^2 \geq \sup_{x \in \mathrm{Dom}\, B, (Bx,x) \leq 1} |(Bx, y)|^2$$

for all $y \in X$. We say that the left hand side of the equation is the form of A on X while the right hand side is that of B.

Also, notice that reflexivity and transitivity of the introduced relation is clear from Lemma 3. To see that the relation is antisymmetric, assume that A and B have the same forms on X. Let $x \in \mathrm{Dom}\, A$ and $y \in \mathrm{Dom}\, B$. Then $(Ax, y) = [J_A^* x, J_A^* y] = [J_B^* x, J_B^* y] = (x, By)$ which means that $B \subseteq A^* = A$ and hence $A = B$.

Now, we establish the maximality of the Friedrichs extension with respect to this partial order.

Theorem 3 *Assume that a is a positive operator from X to X^* with $(ax, x) \geq \gamma \|x\|^2, \gamma > 0$. Then the Friedrichs extension is the largest, positive, self-adjoint extension of a with respect to the partial order introduced above.*

Proof. The completion of $\mathrm{Ran}\, A_F$ with inner product $[A_F x, A_F y]_F := (A_F x, y)$ will be denoted by H_F. Notice first that $\mathrm{Ran}\, a$ is dense in H_F. Indeed, take $A_F x \in H_F$ and let x_n converge to $x = BA_F x \in H$ in the norm of H (recall that H is the completion of $(\mathrm{Dom}\ a, [\cdot, \cdot]))$. Then $ax_n = A_F x_n$ is Cauchy in H_F therefore $x_n \to y$ for some $y \in H_F$. Now, for all $z \in \mathrm{Dom}\, A_F$ we have $[A_F z, A_F x] = (A_F z, x)$ and

$$[A_F z, y] = \lim [A_F z, A_F x_n] = \lim (A_F z, x_n) = (A_F z, x)$$

hence $A_F x = y$. (In the last equality we used the fact that $x_n \to x$ in the norm of X too, due to the positive lower bound of a.)

Next we prove that $\mathrm{Dom}\, a$ is a core for J_F^*. Take an arbitrary element $(x; J^* x) \in X \times H_F$. By the argument above there exists a sequence ax_n such that $ax_n \to J^* x$ in H_F. Now, x_n is Cauchy in H so there is an element $z \in H$ such that $x_n \to z$ in H (and consequently $x_n \to z$ in X). Also, J_F^* is a closed operator therefore $z \in \mathrm{Dom}\, J_F^*$ and $J_F^* z = J_F^* x$. Thus, for all $u \in \mathrm{Dom}\, A_F$ we have $(A_F u, x) = [A_F u, J_F^* x]_F = [A_F u, J_F^* z] = (A_F u, z)$ and A_F is surjective, so this implies $z = x$. Therefore $(x_n; ax_n) \to (x; J^* x)$ in $X \times H_F$.

Now, take an arbitrary extension A of a and compare it to A_F. Define an operator V from H_F to H_A by $\mathrm{Dom}\, V = \mathrm{Ran}\, a$ and $V(ax) = ax$. Then V preserves norms and extends uniquely to an isometry on the whole of H_F. This extension will still be denoted by V. Notice that $J_A^* x = ax$ for all $x \in \mathrm{Dom}\, a$ so we have the inclusion $V(J_F^*|_{\mathrm{Dom}\, a}) \subseteq J_A^*$. Now, $\mathrm{Dom}\, a$ is a core for J_F^* and J_A^* is a closed operator therefore $V J_F^* \subseteq J_A^*$. This means that $\mathrm{Dom}\, J_F^* \subseteq \mathrm{Dom}\, J_A^*$ and $[J_F^* x]_F = [J_A^* x]_A$ so the form of A_F is a restriction of that of A. ∎

Next we turn to the construction of the form sum of two positive self-adjoint operators (cf. [4]).

Here we need a more general notion of closed forms. A positive form $t : D \times D \to \mathbb{C}$ will be called closed if whenever $x_n \subseteq D$ and $x_n \to x$ in X and $t(x_n - x_m, x_n - x_m) \to 0$ then $x \in D$ and $t(x_n - x, x_n - x) \to 0$ (notice that when t has positive lower bound then this definition agrees with the previous one). In this more general context the representation theorem is no longer available (unless X is a Hilbert space), but the form sum construction can be carried out if both forms are closed and at least one of them has positive lower bound.

Assume that A is a positive self-adjoint operator with positive lower bound, and B is an operator associated to a closed form t_B. Assume also that $H_{A,B} := \mathrm{Dom}\, J_A^* \cap \mathrm{Dom}\, t_B$ is dense in X. Then it is easy to see that $(H_{A,B}, t_A + t_B)$ is complete, thus the representation theorem can be applied. The arising positive self-adjoint operator will be called the form sum of A and B, and will be denoted by $A \dot{+} B$. Next we will examine a certain commutation property of the form sum with the help of a factorization similar to that of Lemma 2.

First we give a factorization of the form sum of positive, self-adjoint operators A, B which have positive lower bound. Let $J : H_A \oplus H_B \to X^*$ be the densely defined operator, given by $J(Ax \oplus By) = Ax + By$ for $x \in \mathrm{Dom}\, A, y \in \mathrm{Dom}\, B$. It is obvious that J^* exists and is densely defined. Indeed, it is not difficult to see that

$$\mathrm{Dom}\, J^* = \mathrm{Dom}\, J_A^* \cap \mathrm{Dom}\, J_B^*$$

Let us compute J^* on $\mathrm{Dom}\, A \cap \mathrm{Dom}\, B$. Take z from the common domain of A and B, and let x, y run through $\mathrm{Dom}\, A$ and $\mathrm{Dom}\, B$ respectively. For brevity we will use the unified notation $[\cdot, \cdot]$ for the inner products of the Hilbert spaces H_A, H_B and $H_A \oplus H_B$. Now

$$(J(Ax \oplus By), z) = (Ax + By, z) = [Ax, Az] + [By, Bz] = [Ax \oplus By, Az \oplus Bz],$$

consequently $J^* z = Az \oplus Bz$. Now, if $x \in \mathrm{Dom}\, A \cap \mathrm{Dom}\, B$, then

$$J^{**} J^* x = J^{**}(Ax \oplus Bx) = J(Ax \oplus Bx) = Ax + Bx,$$

which means that $J^{**}J^*$ is an extension of $A + B$.

We claim that $J^{**}J^*$ is nothing else than the form sum $A \dot{+} B$. To see this compute the form determined by the symmetric operator $J^{**}J^*$. Denote by J_A and J_B the factorizing operators for A, B appearing in Lemma 2. Since $J^{**}J^*$ is symmetric, it suffices to show that $A \dot{+} B \subseteq J^{**}J^*$. Take $y \in \mathrm{Dom}\, A \dot{+} B$, then it is straightforward that $y \in \mathrm{Dom}\, J^*$. We should determine $[J^*y, J^*y]$, the following calculation gives an upper estimate.

$$\begin{aligned}
&[J^*y, J^*y] = \\
&\quad \sup\left\{|[Au \oplus Bv, J^*y]|^2 : u \in \mathrm{Dom}\, A, v \in \mathrm{Dom}\, B, (Au,u) + (Bv,v) \le 1\right\} = \\
&\quad \sup\left\{|(Au + Bv, y)|^2 : u \in \mathrm{Dom}\, A, v \in \mathrm{Dom}\, B, (Au,u) + (Bv,v) \le 1\right\} \le \\
&\quad \sup\left\{|[(J_A^* \oplus J_B^*)(u \oplus v), (J_A^* \oplus J_B^*)(y \oplus y)]|^2 : u \in \mathrm{Dom}\, J_A^*, v \in \mathrm{Dom}\, J_B^*, \right. \\
&\qquad\qquad \left. [(J_A^* \oplus J_B^*)(u \oplus v)]^2 \le 1\right\} = \\
&[(J_A^* \oplus J_B^*)(y \oplus y)]^2 = [J_A^*y, J_A^*y] + [J_B^*y, J_B^*y] = (A \dotplus By, y)
\end{aligned}$$

For the reverse inequality set $\lambda = [J_A^*y]^2/([J_A^*y]^2 + [J_B^*y]^2)$ and write:

$$\begin{aligned}
&[J^*y, J^*y] = \\
&\quad \sup\left\{|[Au \oplus Bv, J^*y]|^2 : u \in \mathrm{Dom}\, A, v \in \mathrm{Dom}\, B, (Au,u) + (Bv,v) \le 1\right\} \ge \\
&\quad \sup\left\{|[J_A^*u, J_A^*y] + [J_B^*v, J_B^*y]|^2 : (Au,u) \le \lambda, (Bv,v) \le 1 - \lambda\right\} = \\
&\quad \sup\left\{(|[J_A^*u, J_A^*y]| + |[J_B^*v, J_B^*y]|)^2 : (Au,u) \le \lambda, (Bv,v) \le 1 - \lambda\right\} = \\
&\lambda[J_A^*y]^2 + (1-\lambda)[J_B^*y]^2 + 2\sqrt{\lambda(1-\lambda)}[J_A^*y][J_B^*y] = \\
&\quad [J_A^*y]^2 + [J_B^*y]^2 = (A \dotplus By, y)
\end{aligned}$$

Combining these two estimates, we see that

$$[J^*y, J^*y] = (A \dotplus By, y)$$

for all $y \in \mathrm{Dom}\, A \dotplus B$. This implies that $[J^*x, J^*y] = (A \dotplus Bx, y)$ for all $x, y \in \mathrm{Dom}\, A \dotplus B$, and as consequence we have that $A \dotplus B \subseteq J^{**}J^*$. Thus we have obtained the following theorem.

Theorem 4 *$J^{**}J^* = A \dotplus B$ and it is an extension of $A + B$.*

Now, we observe that commutation with bounded operators is preserved when taking the form sum of two positive, self-adjoint operators. Before all, we characterize this commutation property by means of a bounded operator on an auxiliary Hilbert space.

Let $A : X \to X^*$ be a positive, selfadjoint operator (in the next two lemmas we do not require A to have positive lower bound). Suppose that E is a bounded, linear operator on X which leaves $\mathrm{Dom}\, A$ invariant and satisfies

$$E^*A \subseteq AE. \tag{1}$$

In this case, define $\hat{E}$ on $\mathrm{Ran}\, A \subset H_A$ as $\hat{E}(Ax) := AEx$. This is indeed a definition of a linear operator, as will be seen in the following lemma.

Lemma 4 *$\hat{E}$ is well defined and furthermore is a bounded linear operator on* $\mathrm{Ran}\, A$.

Proof. Fix $x \in \mathrm{Dom}\, A$ and consider the following:

$$[\hat{E}Ax, \hat{E}Ax] = [AEx, AEx] = (AEx, Ex) = (E^*Ax, Ex) = (Ax, E^2x) =$$
$$[Ax, AE^2x] \leq [Ax, Ax]^{\frac{1}{2}}[AE^2x, AE^2x]^{\frac{1}{2}} =$$

By induction, we conclude the following:

$$[\hat{E}Ax, \hat{E}Ax] \leq [Ax, Ax]^{1-\frac{1}{2^n}}[AE^{2^n}x, AE^{2^n}x]^{\frac{1}{2^n}} \leq$$
$$[Ax, Ax]^{1-\frac{1}{2^n}}(Ax, E^{2^{n+1}}x)^{\frac{1}{2^n}} \leq [Ax, Ax]^{1-\frac{1}{2^n}}\|Ax\|^{\frac{1}{2^n}}\|E^{2^{n+1}}x\|^{\frac{1}{2^n}}$$

Now, taking limit in n we obtain

$$[\hat{E}Ax, \hat{E}Ax] \leq [Ax]^2 r(E^2),$$

where $r(E^2)$ denotes the spectral radius of E. This implies that $\hat{E}$ is well-defined and continuous on $\mathrm{Ran}\, A$. ∎

Extend $\hat{E}$ to H_A, and denote this bounded operator also by $\hat{E}$. We connect the self-adjointness of $\hat{E}$ with the commutation property (1) of E in the following (cf. [7]).

Lemma 5 *$\hat{E}$ is a self-adjoint operator in H_A.*

Proof. It suffices to check that $\hat{E}Ax = \hat{E}^*Ax$. So let $x \in \mathrm{Dom}\, A$ be fixed and $y \in \mathrm{Dom}\, A$ arbitrary, then

$$[Ax, \hat{E}Ay] = [Ax, AEy] = (Ax, Ey) = (E^*Ax, y) = (AEx, y) = [\hat{E}Ax, Ay],$$

implying the desired equality. ∎

With the help of the previous arguments, we obtain the theorem on the commutation property of the form sum.

Theorem 5 *Let A, B be two positive, self-adjoint operators with positive lower bound, and assume that a bounded, linear operator E is given on X, such that* (1) *is satisfied together with a similar condition concerning B. Then for the form sum $A \dot{+} B$, we have*

$$E^*(A \dot{+} B) \subseteq (A \dot{+} B)E.$$

Proof. Denote the bounded, self-adjoint operators resulting from the above construction on H_A and H_B by $\hat{E}_A$ and $\hat{E}_B$ respectively. Consider the bounded, self-adjoint operator $\hat{E}_A \oplus \hat{E}_B$ on the Hilbert space $H_A \oplus H_B$. Then $E^*J \subseteq J(\hat{E}_A \oplus \hat{E}_B)$ and $(\hat{E}_A \oplus \hat{E}_B)J^* \subseteq J^*E$, since for any $x \in \mathrm{Dom}\, A, y \in \mathrm{Dom}\, B$

$$J\left(\hat{E}_A \oplus \hat{E}_B\right)(Ax \oplus By) = J(AEx \oplus BEy) = AEx + BEy =$$
$$E^*Ax + E^*By = E^*J(Ax \oplus By),$$

and from this, using the continuity of E^*, it follows that

$$\left(\hat{E}_A \oplus \hat{E}_B\right) J^* \subseteq \left[J\left(\hat{E}_A \oplus \hat{E}_B\right)\right]^* \subseteq (E^* J)^* = J^* E.$$

We use these two inclusions and the boundedness of the self-adjoint operator $\hat{E}_A \oplus \hat{E}_B$ to conclude

$$E^* J^{**} \subseteq (J^* E)^* \subseteq \left[\left(\hat{E}_A \oplus \hat{E}_B\right) J^*\right]^* = J^{**}\left(\hat{E}_A \oplus \hat{E}_B\right) \text{ and}$$
$$E^*\left(A \dot{+} B\right) = E^* J^{**} J^* \subseteq J^{**}\left(\hat{E}_A \oplus \hat{E}_B\right) J^* \subseteq J^{**} J^* E = \left(A \dot{+} B\right) E.$$

■

Finally we observe the relation between the spectra of $\hat{E} = \hat{E}_A$ and that of E. (Here again the requirement that A has psoitive lower bound is not necessary.)

Theorem 6 *The spectrum $\sigma(\hat{E})$ is contained in $\sigma(E) \cap \mathbb{R}$.*

Proof. Since $\hat{E}$ is self-adjoint it is clear that $\sigma(\hat{E}) \subseteq \mathbb{R}$. On the other hand, take any real λ from the resolvent set of E and $x \in \operatorname{Dom} A$, then

$$A(E - \lambda I)x = (E - \lambda I)^* Ax, \text{ hence } A(E - \lambda I)^{-1}x = \left[(E - \lambda I)^{-1}\right]^* Ax$$

which means that we can define the operator $[(E-\lambda I)^{-1}]\widehat{}$. A short computation gives that

$$\widehat{\left[(E - \lambda I)^{-1}\right]} = \left(\hat{E} - \lambda \hat{I}\right)^{-1},$$

indeed for $x \in \operatorname{Dom} A$

$$\widehat{\left[(E - \lambda I)^{-1}\right]}\left(\hat{E} - \lambda \hat{I}\right) Ax = A\,(E - \lambda I)^{-1}\,(E - \lambda I)\,x = Ax, \text{ and}$$
$$\left(\hat{E} - \lambda \hat{I}\right) \widehat{\left[(E - \lambda I)^{-1}\right]} Ax = A\,(E - \lambda I)\,(E - \lambda I)^{-1}\,x = Ax.$$

This proves the statement. ■

4 Application of the results

Covariance operators.

Consider a probability measure space $\langle \Omega, \mathcal{A}, \mu \rangle$, and let $\xi : \Omega \to X$ a random variable i.e. a weakly measurable function. Suppose that ξ possesses a weak expectation, in other words

$$\mathrm{E}\,\xi := \int_\Omega \xi \,\mathrm{d}\mu$$

exists as a Pettis integral. Note that if X is reflexive, according to Dunford and Gelfand, this is equivalent to requiring the existence of

$$\int_\Omega f(\xi)\,\mathrm{d}\mu$$

for all $f \in X^*$. Further, we make assumptions on the second moments, so suppose that the set

$$D = \left\{ f : f \in X^*, \int_\Omega |f(\xi)|^2 \,\mathrm{d}\mu < +\infty \right\}$$

is dense in X^*.

As an example, take $X = \ell_2$, $\Omega = \{\omega_n : n = 1, 2, \dots\}$ and $\mu(\{\omega_n\}) = ce^{-(3/2)n}$ with a suitable constant c. Setting $\xi(\omega_n)_k = n^k/k!$, it is easy to compute that, in this case, $D \neq X^*$ is dense.

In the sequel we assume that $\mathrm{E}\,\xi = 0$, since we could take $\xi - \mathrm{E}\,\xi$ instead of ξ. Define the sesquilinear form

$$t(f, g) = \mathrm{E}\ (f(\xi)\bar{g}(\xi))$$

for $f, g \in D$.

Theorem 7 *t is a positive, closed, sesquilinear form on $D \times D$.*

Proof. Positivity is trivial. Suppose that $f_n \in D$ converges to $f \in X^*$ and $\mathrm{E}\ |f_n(\xi) - f_m(\xi)|^2 \to 0$, then $f_n(\xi)$ has a limit $g \in \mathcal{L}_2(\Omega, \mu)$, and moreover g and $f(\xi)$ conincide almost everywhere, hence $\mathrm{E}\,|f(\xi)|^2 < +\infty$, implying $f \in D$ and $\mathrm{E}\ |f_n(\xi) - f(\xi)|^2 \to 0$. ∎

If t possesses a positive lower bound and X is reflexive, then the application of Theorem 1 provides a representing, self-adjoint operator A from X^* to $X^{**} = X$, which is called the covariance operator of the random variable ξ (cf. [11]). Note that if X is a Hilbert space then the original version of the representation theorem provides the covariance operator of ξ associated to the closed form t (even if t has lower bound 0).

One expects a characterization of the covariance operator of the sum of independent random variables. In view of our result the following theorem is quite straightforward.

Theorem 8 *If ξ and η are independent random variables with covariance operators A and B respectively. Then the covariance operator of $\xi + \eta$ is $A \dot{+} B$.*

Elliptic operators.

This is a classical application of the Friedrichs extension (see e.g. [2]). Take $X = L_p(\Omega), 1 \leq p < +\infty$ where Ω is a bounded domain with smooth boundary in $\mathbb{R}^n$. Define the operator A from $L_p(\Omega)$ to $L_q(\Omega)$ by Dom $A = C_0^\infty(\Omega)$ and

$$Af = -\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}(a_{ik}\frac{\partial f}{\partial x_k}) + bf$$

where $a_{ik} \in C^1(\Omega)$, $b \in L^1_{loc}(\Omega)$, $b \geq 0$ and

$$\sum_{i,k=1}^{n} a_{ik}(x)\beta_i\overline{\beta_k} \geq \gamma \sum_{i}^{n} |\beta_i|^2, \gamma > 0$$

everywhere in Ω (uniform ellipticity). In this case we have

$$\begin{aligned}(Af, f) &= \int_\Omega \left(-\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial f}{\partial x_k}\right) + bf \right) \overline{f}\,\mathrm{d}x \\ &= \int_\Omega \left(\sum_{i,k=1}^{n} a_{ik}\frac{\partial f}{\partial x_i}\frac{\overline{\partial f}}{\partial x_k} + b|f|^2 \right) \mathrm{d}x \geq \gamma \int_\Omega \sum_{i=1}^{n} \left|\frac{\partial f}{\partial x_i}\right|^2 \mathrm{d}x.\end{aligned}$$

Now, for $p \leq 2n/(n-2)$ we have

$$\int_\Omega \sum_{i=1}^{n} \left|\frac{\partial f}{\partial x_i}\right|^2 \mathrm{d}x \geq c\|f\|_p^2, \quad c > 0$$

by the Sobolev imbedding theorem (see e.g. [1] pp. 95-99). Thus A has positive lower bound. The Friedrichs extension of A is surjective, and this means that the equation

$$-\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial f}{\partial x_k}\right) + bf = g$$

has a weak solution for every $g \in L_q(\Omega)$ whenever $q \geq 2n/(n+2)$.

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