On the relation of closed forms and Trotter's product formula

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Abstract

The aim of this paper is to give a characterization in Hilbert spaces of the generators of C_0-semigroups associated with closed, sectorial forms in terms of the convergence of a generalized Trotter's product formula. In the course of the proof of the main result we also present a similarity result which can be of independent interest: for any unbounded generator A of a C_0-semigroup e^{tA} it is possible to introduce an equivalent scalar product on the space, such that e^{tA} becomes non-quasi-contractive with respect to the new scalar product.

Key words: closed,sectorial forms, Trotter-Kato product formula, quasi-contractivity

1 Introduction

Let H denote a complex Hilbert space, A the generator of a C_0-semigroup e^{tA} on H, and P a bounded projection. The convergence of the generalized Trotter's product formula

$$\lim_{n\to\infty}(e^{\frac{t}{n}A}P)^n \tag{1}$$

was first studied (as a corollary of the main Theorem) in [5]. It was shown that (1) converges strongly for all $t > 0$ whenever $-A$ is a non-negative self-adjoint operator, and P is an orthogonal projection. More generally, the Addendum in [5] implies that the same result is true whenever $-A$ is associated with a closed sectorial form (where the vertex of the sector is allowed to be any real number ω, i.e. $S_{\phi,\omega} := \{z \in \mathbb{C} : -\phi < \arg\,(z-\omega) < \phi\}$, $\omega \in \mathbb{R}$, $\phi \in (0, \frac{\pi}{2})$).

The convergence of (1) was then studied in more general settings in [1] and [3] where further convergence results, motivating examples, and some counterexamples were given. The main result of this paper is to prove the converse of Kato's result, i.e. that the strong convergence of (1) for all orthogonal projections P, in fact, characterizes generators A such that $-A$ is associated with a closed sectorial form. To be more precise we recall the following result (see [5] Addendum, and [3] Theorem 1 and 4.):

Theorem 1 *Let A be the generator of a C_0-semigroup e^{tA} on a Hilbert space H. Consider the following statements:*

(i) A is bounded.

(ii) $-A$ is associated with a densely-defined, closed, sectorial form a on H.

(iii) The formula $(e^{\frac{t}{n}A}P)^n x$ converges for all projections $P \in \mathcal{L}(H)$, and all $x \in H$ and $t > 0$.

(iv) The formula $(e^{\frac{t}{n}A}P)^n x$ converges for all orthogonal projections $P \in \mathcal{L}(H)$, and all $x \in H$ and $t > 0$.

The following implications hold: (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv).

We will show in Section 2 that the converse implications also hold. In the course of the proof we will need an auxillary result, given in Theorem 2, which can be regarded as a complement of [4], and is of independent interest. Namely, we show that whenever the generator A of the semigroup e^{tA} is unbounded, it is possible to introduce an equivalent scalar product $(\ ,\ )_0$ on H such that e^{tA} is non-quasi-contractive with respect to $(\ ,\ )_0$.

2 Main Result

In order to prove our main result [Theorem 3], first we need to characterize the class of generators A on H, such that the C_0-semigroup e^{tA} is quasi-contractive for *every* equivalent scalar product $(\ ,\ )_0$ on H. The characterization is provided by

Theorem 2 *Let A be the generator of a C_0-semigroup e^{tA} on a Hilbert space H. The following are equivalent:*

(i) A is bounded.

(ii) The semigorup e^{tA} is quasi-contractive for every equivalent scalar product $(\ ,\ )_0$ on H.

(iii) For every equivalent scalar product $(\ ,\ )_0$ on H there exists $K_0 \in \mathbb{R}$ such that for every vector $x \in D(A)$, $(x,x)_0 = 1$ implies $\mathrm{Re}\,(Ax,x)_0 \leq K_0$.
Proof. The implications (ii) $\Leftrightarrow$ (iii) are consequences of the Lumer-Phillips theorem (see e.g. [2], Proposition 3.23.). The implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are trivial. It remains to prove (iii) $\Rightarrow$ (i). We will need the following
Definition 1 *Let $T \in \mathcal{L}(H)$ be an injective operator, and $x \in H$, $\|x\| = 1$, and $0 < \delta \leq 1$. We say that x is a δ-quasi-eigenvector of T if*

$$\delta \leq \frac{|(x,Tx)|}{\|Tx\|} \leq 1 \tag{2}$$

Note, that a 1-quasi-eigenvector is, in fact, an eigenvector of T.
Now, let $0 < \delta < 1$ be fixed. We prove the implication (iii) $\Rightarrow$ (i) by contradiction. Assume, therefore, that $A \notin \mathcal{L}(H)$, and also, by rescaling, that $A^{-1} =: T \in \mathcal{L}(H)$. Assume, furthermore, that a sequence $(h_n) \subset H$ is given with the following properties:
(a) $\|h_n\| = 1$ for all $n \geq 1$.
(b) $\{h_k,\ Th_k\} \perp \{h_j,\ Th_j\}$ for all $k \neq j$.
(c) $\lim_{n\to\infty} \|Th_n\| = 0$
(d) For every $n \geq 1$ the vector h_n is *not* a δ-quasi-eginvector of T.
We construct an equivalent scalar product $(\ ,\ )_0$ on H with the help of the sequence h_n.
Let $H_n = \mathrm{span}\{h_n,\ Th_n\}$. Note, that H_n is 2-dimensional because h_n is not an eigenvector of T.
Let $Th_n = c_{1,n}h_n + c_{2,n}h_n^\perp$, where $\|h_n^\perp\| = 1$. Note that

$$\frac{|c_{1,n}|^2}{|c_{1,n}|^2 + |c_{2,n}|^2} < \delta^2 \quad \text{and} \quad \frac{|c_{2,n}|^2}{|c_{1,n}|^2 + |c_{2,n}|^2} > 1 - \delta^2$$

Hence,

$$\frac{|c_{1,n}|}{|c_{2,n}|} < \frac{\delta}{\sqrt{1-\delta^2}} \quad \text{and} \quad \frac{|c_{2,n}|}{\|Th_n\|} > \sqrt{1-\delta^2}$$

Define $Q_n \in \mathcal{L}(H_n)$ by

$$\begin{aligned} Q_n h_n &:= h_n + \overline{L}_n h_n^\perp \\ Q_n h_n^\perp &:= L_n h_n + (|L_n|^2 + 1) h_n^\perp \end{aligned}$$

where $|L_n| = 2\frac{\delta}{\sqrt{1-\delta^2}}$ and $\overline{L_n c_{2,n}} > 0$ for all $n \geq 1$. It is clear that $Q_n = Q_n^* \geq 0$, $Q_n^{-1} \in \mathcal{L}(H_n)$, and $\|Q_n\|_{H_n} \leq K$, $\|Q_n^{-1}\|_{H_n} \leq K$ for some universal constant K (not depending on n). Define $Q \in \mathcal{L}(H)$ by

$$Q := Q_1 \oplus Q_2 \oplus \cdots \bigoplus I_{(H_1 \oplus H_2 \oplus \ldots)^\perp}$$

It is easy to see that Q is well-defined, $Q \in \mathcal{L}(H)$, $Q = Q^* \geq 0$, and $Q^{-1} \in \mathcal{L}(H)$. This means that Q defines an equivalent scalar product on H by $(x,y)_0 := (x,Qy)$.

Now, let $x_n := \frac{Th_n}{\|Th_n\|}$. Then

$$\text{Re } (Ax_n, x_n)_0 = \frac{1}{\|Th_n\|^2}\text{Re } (h_n, QTh_n) = \frac{1}{\|Th_n\|^2}\text{Re } (h_n, c_{1,n}h_n + c_{2,n}L_n h_n) =$$
$$= \frac{1}{\|Th_n\|^2}(\text{Re } c_{1,n} + \overline{c_{2,n}L_n}) \geq \frac{1}{\|Th_n\|^2}\frac{\delta}{\sqrt{1-\delta^2}}|c_{2,n}| \geq \frac{1}{\|Th_n\|}\delta \to +\infty$$

Let $y_n := \frac{x_n}{\|x_n\|_0}$. Then Re $(Ay_n, y_n)_0 \to +\infty$ still holds due to the equivalence of the scalar products $(\ ,\ )$ and $(\ ,\ )_0$.
In order to complete the proof of the theorem it remains to construct the sequence h_n with the required properties. The construction is carried out in several steps.
Step 1. We construct an orthonormal sequence $(e_n) \subset H$, such that $\lim_{n\to\infty} \|Te_n\| = 0$.
Take the polar decomposition $T = UT_1$ of T, where U is unitary and $T_1 = T_1^* \geq 0$. It is clear from the spectral theorem that there exists an orthonormal sequence $(e_n) \subset H$ such that $\lim_{n\to\infty} \|T_1 e_n\| = 0$ (otherwise T_1 and T would be invertible, contrary to our assumption). Note, also, that $\|T_1 e_n\| = \|Te_n\|$ for all $n \in \mathbb{N}$, therefore $\lim_{n\to\infty} \|Te_n\| = 0$ as required.
Step 2. We construct an orthonormal sequence $(f_n) \subset H$ such that $\lim_{n\to\infty} \|Tf_n\| = 0$ and $f_{n+1} \perp \{f_1, Tf_1, ...f_n, Tf_n\}$.
We obtain the sequence (f_n) by induction, with the help of the sequence (e_n). Take an index i_1 such that $\|Te_{i_1}\| \leq 1$, and let $f_1 := e_{i_1}$. Assume now that $f_1, f_2, \ldots, f_n$ are already given such that

$$\|f_j\| = 1, \quad f_j \perp \{f_k, Tf_k\}, \quad \|Tf_j\| \leq \frac{1}{\sqrt{j}}$$

and $f_j \in \text{span}\{e_1, e_2, \ldots, e_{l_n}\}$, for all $1 \leq j, k \leq n$, $k < j$, and l_n is an index depending on n only.
Let $H_n := \text{span}\{Tf_1, Tf_2, \ldots, Tf_n\}$. Take indices $j_1, \ldots j_{n+1}$ such that $j_k > l_n$ and $\|Te_{j_k}\| \leq \frac{1}{n+1}$ for all $1 \leq k \leq n+1$. The subspace H_n is at most n-dimensional, therefore there exists a non-trivial linear combination

$$f_{n+1} := \sum_{k=1}^{n+1} \lambda_k e_{j_k}$$

such that $\|f_{n+1}\| = 1$ and $f_{n+1} \perp H_n$.
It is clear, by construction, that $f_{n+1} \perp \{f_1, Tf_1, ...f_n, Tf_n\}$. Furthermore,

$$\|Tf_{n+1}\| \leq \frac{1}{n+1}\sum_{k=1}^{n+1}|\lambda_k| \leq \sqrt{\frac{\sum_{k=1}^{n+1}|\lambda_k|^2}{n+1}} = \frac{1}{\sqrt{n+1}}$$

Step 3. We construct an orthonormal sequence $(g_n) \subset H$ such that $\lim_{n\to\infty} \|Tg_n\| = 0$ and $\{g_j, Tg_j\} \perp \{g_k, Tg_k\}$ for all $j \neq k$.
We obtain the sequence (g_n) by induction, with the help of the sequence (f_n).

Let $g_1 = f_1$. Assume now that $g_1, g_2, \ldots, g_n$ are already given such that

$$\|g_j\| = 1, \quad \{g_j, Tg_j\} \perp \{g_k, Tg_k\}, \quad \|Tg_j\| \leq \frac{1}{\sqrt{2j-1}}$$

and $g_j \in \operatorname{span}\{f_1, f_2, \ldots, f_{b_n}\}$, for all $1 \leq j \neq k \leq n$, and b_n is an index depending on n only.
Let $G_n := \operatorname{span}\{g_1, Tg_1, g_2, Tg_2, \ldots, g_n, Tg_n\}$. Take indices $m_1, \ldots m_{2n+1}$ such that $m_k > b_n$ and $\|Tf_{m_k}\| \leq \frac{1}{2n+1}$ for all $1 \leq k \leq 2n+1$. The subspace G_n is at most $2n$-dimensional, therefore there exists a non-trivial linear combination

$$g_{n+1} := \sum_{k=1}^{2n+1} \mu_k f_{m_k}$$

such that $\|g_{n+1}\| = 1$ and $Tg_{n+1} \perp G_n$.
It is clear, by construction, that $\{g_{n+1}, Tg_{n+1}\} \perp \{g_1, Tg_1, ...g_n, Tg_n\}$. Furthermore,

$$\|Tg_{n+1}\| \leq \frac{1}{2n+1} \sum_{k=1}^{2n+1} |\mu_k| \leq \sqrt{\frac{\sum_{k=1}^{2n+1} |\mu_k|^2}{2n+1}} = \frac{1}{\sqrt{2(n+1)-1}}$$

Step 4. We construct the orthonormal sequence (h_n) with the properties stated at the beginning of the proof.
We obtain the sequence (h_n) by induction, with the help of the sequence (g_n). Take an index r_1 such that $\|Tg_{r_1}\| \leq \frac{\delta^2}{10}\|Tg_1\|$. Let

$$h_1 := \frac{\delta}{2} g_1 + \sqrt{1 - \frac{\delta^2}{4}} g_{r_1}$$

We need to prove that h_1 is not a δ-quasi-eigenvector of T. It is clear that

$$1 \geq \|Th_1\| \geq \left(\frac{\delta}{2} - \frac{\delta^2}{10}\sqrt{1 - \frac{\delta^2}{4}} \right) \|Tg_1\|$$

Also,

$$|(h_1, Th_1)| = |(g_1, Tg_1) + (g_{r_1}, Tg_{r_1})| \leq \left(\frac{\delta^2}{4} + (1 - \frac{\delta^2}{4}) \frac{\delta^2}{10} \right) \|Tg_1\|$$

Combining these two ineqalities a simple calculation shows that $\frac{|(h_1, Th_1)|}{\|Th_1\|} < \delta$, as required.
Assume now that vectors $h_1, \ldots, h_n$ are already given, such that h_j is not a δ-quasi-eigenvector of T,

$$\|h_j\| = 1, \quad \{h_j, Th_j\} \perp \{h_k, Th_k\}, \quad \|Th_j\| \leq \frac{1}{\sqrt{j}}$$

and $h_j \in \operatorname{span}\{g_1, g_2, \ldots, g_{a_n}\}$, for all $1 \le j \ne k \le n$, and a_n is an index depending on n only. Take indices p_1, p_2, such that $p_1, p_2 > a_n$ and $\|Tg_{p_1}\| \le \frac{1}{\sqrt{n+1}}$, and $\|Tg_{p_2}\| \le \frac{\delta^2}{10}\|Tg_{p_1}\|$. Let

$$h_{n+1} := \frac{\delta}{2} g_{p_1} + \sqrt{1 - \frac{\delta^2}{4}} g_{p_2}$$

It is clear that $\|Th_{n+1}\| \le \frac{1}{\sqrt{n+1}}$, and it can be shown as above that h_{n+1} is not a δ-quasi-eigenvector of T. Hence, the sequence (h_n) satisfies all requirements, and the proof is complete. $\Box$

The author conjectures that a result corresponding to Theorem 2 holds also in Banach spaces. Namely, whenever A is not bounded it should be possible to introduce an equivalent norm on the space such that e^{tA} is not quasi-contractive with respect to the new norm. This problem, however, remains open.

Now we present the main result of the paper.

Theorem 3 *Let A be the generator of a C_0-semigroup e^{tA} on a Hilbert space H. Consider the following statements.*

(i) A is bounded.

(ii) $-A$ is associated with a densely-defined, closed, sectorial form a on H.

(iii) The formula $(e^{\frac{t}{n}A}P)^n x$ converges for all projections $P \in \mathcal{L}(H)$, and all $x \in H$ and $t > 0$.

(iv) The formula $(e^{\frac{t}{n}A}P)^n x$ converges for all orthogonal projections $P \in \mathcal{L}(H)$, and all $x \in H$ and $t > 0$.

The following implications hold: (i) $\Leftrightarrow$ (iii), (ii) $\Leftrightarrow$ (iv).

Proof. The implication (i) $\Rightarrow$ (iii) was proved in [3], while the implication (ii) $\Rightarrow$ (iv) is a consequence of [5], Addendum (see also [3], Theorem 4).
We prove the implication (iii) $\Rightarrow$ (i) by contradiction.
Assume first that the semigroup e^{tA} is not quasi-contractive. By the Lumer-Phillips theorem this is equivalent to the fact that the numerical range of A is not contained in any left half-plane.
We construct an element $g \in H$ such that $\|g\| = 1$, and

$$\lim_{n\to\infty} (e^{\frac{1}{n}A} P_g)^n g$$

does not exist, where P_g denotes the one-dimensional projection onto the subspace spanned by g. The vector g will be given as

$$g := \frac{\lim_{k\to\infty} g_k}{\|\lim_{k\to\infty} g_k\|}$$

where (g_k) denotes a convergent sequence in H to be constructed in the sequel. Let $g_1 \in D(A)$, such that $\|g_1\| = 1$. First, we show that

$$\lim_{n\to\infty} (e^{\frac{1}{n}A} P_{g_1})^n g_1 = e^{(Ag_1, g_1)} g_1$$

Indeed,

$$(e^{\frac{1}{n}A} P_{g_1})^n g_1 = e^{\frac{1}{n}A} (P_{g_1} e^{\frac{1}{n}A} P_{g_1})^{n-1} g_1 = e^{\frac{1}{n}A} (P_{g_1} e^{\frac{1}{n}A} P_{g_1} g_1,\ g_1)^{n-1} g_1$$

and

$$\lim_{n\to\infty} (P_{g_1} e^{\frac{1}{n}A} P_{g_1} g_1,\ g_1)^{n-1} = e^{(Ag_1,\ g_1)}$$

because

$$\lim_{n\to\infty} \frac{(P_{g_1} e^{\frac{1}{n}A} P_{g_1} g_1,\ g_1) - 1}{1/n} = \lim_{n\to\infty} \left(\frac{(e^{\frac{1}{n}A} - I) g_1}{1/n},\ g_1 \right) = (Ag_1,\ g_1)$$

Now, choose g_1 such that Re $(Ag_1, g_1) \geq 1$ holds also.
Let $\varepsilon > 0$ be fixed. Take an index n_1 so large that

$$\| \left(e^{\frac{1}{n_1}A} P_{g_1} \right)^{n_1} g_1 - e^{(Ag_1, g_1)} g_1 \| < \varepsilon$$

It is clear from standard continuity arguments that there exists a $\delta_1 > 0$, such that for all $h \in B(g_1, \delta_1)$ we have

$$\| \left(e^{\frac{1}{n_1}A} P_{\frac{h}{\|h\|}} \right)^{n_1} \frac{h}{\|h\|} - e^{(Ag_1, g_1)} g_1 \| < 2\varepsilon$$

Without loss of generality we can assume that $\delta_1 < \frac{1}{2}$.
Now assume, that vectors $g_1, g_2, \ldots, g_k$, and positive numbers $\delta_1, \delta_2, \ldots, \delta_k$, and indices $n_1, n_2, \ldots, n_k$ are already given with the properties that:

$$g_j \in D(A), \quad \text{Re } (Ag_j, g_j) \geq j$$

and

$$\| \left(e^{\frac{1}{n_j}A} P_{\frac{h}{\|h\|}} \right)^{n_j} \frac{h}{\|h\|} - e^{(A \frac{g_j}{\|g_j\|}, \frac{g_j}{\|g_j\|})} \frac{g_j}{\|g_j\|} \| < 2\varepsilon$$

for all $1 \leq j \leq k$ and all $h \in B(g_j, \delta_j)$. Assume, furthermore, that

$$\|g_{j+1} - g_j\| < \min \left\{ \frac{\delta_1}{2^j}, \frac{\delta_2}{2^{j-1}}, \ldots \frac{\delta_j}{2} \right\}$$

for all $1 \leq j \leq k-1$.

The numerical range of A is not bounded from the right, hence there exists a vector $f \in D(A)$ such that

$$\|f\| < \min \left\{ \frac{1}{\|Ag_k\|}, \frac{\delta_1}{2^k}, \frac{\delta_2}{2^{k-1}}, \dots \frac{\delta_k}{2} \right\}$$

and Re $(Af, f) \geq 2$. Let $f_k := e^{i\alpha} f$ with suitable α such that Re $(Af_k, g_k) \geq 0$. Let

$$g_{k+1} := g_k + f_k$$

Then

$$\begin{aligned} \text{Re } (Ag_{k+1}, g_{k+1}) &= \text{Re } (Ag_k, g_k) + \text{Re } (Ag_k, f_k) + \\ +\text{Re } (Af_k, g_k) + \text{Re } (Af_k, f_k) &\geq k + (-1) + 0 + 2 = k + 1 \end{aligned}$$

Furthermore, we have

$$\lim_{n\to\infty} (e^{\frac{1}{n}A} P_{\frac{g_{k+1}}{\|g_{k+1}\|}})^n \frac{g_{k+1}}{\|g_{k+1}\|} = e^{(A\frac{g_{k+1}}{\|g_{k+1}\|}, \frac{g_{k+1}}{\|g_{k+1}\|})} \frac{g_{k+1}}{\|g_{k+1}\|}$$

Take an index n_{k+1} so large that $n_{k+1} > n_k$ and

$$\| \left(e^{\frac{1}{n_{k+1}}A} P_{\frac{g_{k+1}}{\|g_{k+1}\|}} \right)^{n_{k+1}} \frac{g_{k+1}}{\|g_{k+1}\|} - e^{(A\frac{g_{k+1}}{\|g_{k+1}\|}, \frac{g_{k+1}}{\|g_{k+1}\|})} \frac{g_{k+1}}{\|g_{k+1}\|} \| < \varepsilon$$

It is clear from standard continuity arguments that there exists a $\delta_{k+1} > 0$, such that for all $h \in B(g_{k+1}, \delta_{k+1})$ we have

$$\| \left(e^{\frac{1}{n_{k+1}}A} P_{\frac{h}{\|h\|}} \right)^{n_{k+1}} \frac{h}{\|h\|} - e^{(A\frac{g_{k+1}}{\|g_{k+1}\|}, \frac{g_{k+1}}{\|g_{k+1}\|})} \frac{g_{k+1}}{\|g_{k+1}\|} \| < 2\varepsilon$$

It is clear, by construction, that the sequence g_k converges in H. Let

$$h := \lim_{k\to\infty} g_k \quad \text{and} \quad g := \frac{h}{\|h\|}$$

Recall, that $\|g_1\| = 1$ and $\delta_1 < \frac{1}{2}$, therefore $\frac{1}{2} < \|g_k\| < \frac{3}{2}$ for all $k \geq 1$. It is also clear, by construction, that $h \in B(g_k, \delta_k)$ for all $k \geq 1$. Hence, for all $k \geq 1$ we have

$$\| \left(e^{\frac{1}{n_k}A} P_g \right)^{n_k} g - e^{(A\frac{g_k}{\|g_k\|}, \frac{g_k}{\|g_k\|})} \frac{g_k}{\|g_k\|} \| < 2\varepsilon$$

Notice, that

$$\|e^{(A\frac{g_k}{\|g_k\|}, \frac{g_k}{\|g_k\|})} \frac{g_k}{\|g_k\|}\| = e^{\frac{1}{\|g_k\|^2}\text{Re } (Ag_k, g_k)} > e^{\frac{1}{4}k}$$

This means that (the norm of) the sequence $(e^{\frac{1}{n}A} P_g)^n g$ does not converge. Now, assume only that $A \notin \mathcal{L}(H)$. Introduce, by Theorem 2, an equivalent scalar product $(x, y)_0 := (x, Qy)$ on H, such that the semigroup e^{tA} is not

quasi-contractive with respect to $(\ ,\ )_0$. Take an orthogonal projection P_g (with respect to the scalar product $(\ ,\ )_0$), such that $(e^{\frac{1}{n}A}P_g)^n g$ does not converge. Then, P_g is a bounded (possibly non-orthogonal) projection with respect to the original scalar product $(\ ,\ )$, such that $(e^{\frac{1}{n}A}P_g)^n g$ does not converge. This proves the implication (iii) $\Rightarrow$ (i).
The implication (iv) $\Rightarrow$ (ii) is also proved by contradiction.
Asume, that the numerical range of A is not contained in any sector

$$\Sigma_{\phi,\omega} := \{z \in \mathbb{C} : \frac{\pi}{2} + \phi < \arg\ (z - \omega) < \frac{3}{2}\pi - \phi\}$$

with $\omega \in \mathbb{R}$, $\phi \in (0, \frac{\pi}{2})$. There are two cases to consider.
If the semigroup e^{tA} is not quasi-contractive , then, by the arguments above, there exists a vector $g \in H$, such that $\|g\| = 1$ and $(e^{\frac{1}{n}A}P_g)^n g$ does not converge.
If the semigroup e^{tA} is quasi-contractive then, by rescaling, we can assume that Re $(Ax, x) \leq -1$ for all $x \in D(A)$, $\|x\| = 1$.
We construct an element $g \in H$ such that $\|g\| = 1$, and $\lim_{n\to\infty}(e^{\frac{1}{n}A}P_g)^n g$ does not exist, where P_g denotes the one-dimensional projection onto the subspace spanned by g. The vector g will be given as

$$g := \frac{\lim_{k\to\infty} g_k}{\|\lim_{k\to\infty} g_k\|}$$

where (g_k) denotes a convergent sequence in H to be constructed in the sequel.
Take an arbitrary vector $g_1 \in D(A)$, $\|g_1\| = 1$. Let $(Ag_1, g_1) =: a_1 + b_1 i$. We know that

$$\lim_{n\to\infty} \left(e^{\frac{1}{n}A}P_{g_1}\right)^n g_1 = e^{(Ag_1,g_1)}g_1$$

Let $\varepsilon > 0$, and $\rho > 0$ be fixed. Take an index n_1 so large that

$$\| \left(e^{\frac{1}{n_1}A}P_{g_1}\right)^{n_1} g_1 - e^{(Ag_1,g_1)}g_1\| < \varepsilon$$

It is clear from standard continuity arguments that there exists a $\delta_1 > 0$, such that for all $h \in B(g_1, \delta_1)$ we have

$$\| \left(e^{\frac{1}{n_1}A}P_{\frac{h}{\|h\|}}\right)^{n_1} \frac{h}{\|h\|} - e^{(Ag_1,g_1)}g_1\| < 2\varepsilon$$

Without loss of generality we can assume that $\delta_1 < \frac{1}{2}$.
Now assume, that vectors $g_1, g_2, \ldots, g_k$, and positive numbers $\delta_1, \delta_2, \ldots, \delta_k$, real numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$, and indices $n_1, n_2, \ldots, n_k$ are already given with the following properties: for all $1 \leq j \leq k$ we have $|\varepsilon_j| < \rho$,

$$g_j \in D(A),\quad (A\frac{g_j}{\|g_j\|}, \frac{g_j}{\|g_j\|}) = a_j + (\varepsilon_j + b_1 + (j-1)\pi)i$$

(note that $\varepsilon_1 = 0$), where $a_1 - 1 < a_j \leq -1$, and

$$\| \left(e^{\frac{1}{n_j}A} P_{\frac{h}{\|h\|}} \right)^{n_j} \frac{h}{\|h\|} - e^{(A\frac{g_j}{\|g_j\|}, \frac{g_j}{\|g_j\|})} \frac{g_j}{\|g_j\|} \| < 2\varepsilon$$

for all $h \in B(g_j, \delta_j)$. Assume, furthermore, that

$$\|g_{j+1} - g_j\| < \min \left\{ \frac{\delta_1}{2^j}, \frac{\delta_2}{2^{j-1}}, \dots \frac{\delta_j}{2} \right\}$$

for all $1 \leq j \leq k-1$.
Now, we construct the vector g_{k+1}. The numerical range of A is not contained in any sector, therefore there exists a sequence $(x_j) \subset D(A)$ such that, $\lim_{j\to\infty} \|x_j\| = 0$ and

$$\mathrm{Im}\, \frac{(Ax_j, x_j)}{\|g_k\|^2} = \pi \quad \text{and} \quad \frac{\mathrm{Re}\ (Ax_j, x_j)}{\|g_k\|^2} < \frac{a_k - (a_1 - 1)}{2}$$

Take $y_j := e^{i\alpha_j} x_j$ with suitable α_j such that $(Ay_j, g_k) \geq 0$ real. Then

$$\frac{(A(g_k + y_j), g_k + y_j)}{\|g_k\|^2} = \frac{(Ag_k, g_k)}{\|g_k\|^2} + \frac{(Ag_k, y_j)}{\|g_k\|^2} + \\ + \frac{(Ay_j, g_k)}{\|g_k\|^2} + \frac{(Ay_j, y_j)}{\|g_k\|^2} =: c_j + d_j i$$

The real part c_j of this expression satisfies

$$c_j > (a_1 - 1) + (\frac{a_k - (a_1 - 1)}{2}) - \frac{|(Ag_k, y_j)|}{\|g_k\|^2}$$

for all $j \geq 1$. For the imaginary part d_j, we have

$$\lim_{j\to\infty} d_j = \varepsilon_k + b_1 + k\pi$$

This means that for large j we have $\|y_j\| < \min \{\frac{\delta_1}{2^k}, \frac{\delta_2}{2^{k-1}}, \dots \frac{\delta_k}{2}\}$, and

$$\frac{\mathrm{Re}(A(g_k + y_j), g_k + y_j)}{\|g_k + y_j\|^2} > a_1 - 1$$

and

$$\frac{\mathrm{Im}(A(g_k + y_j), g_k + y_j)}{\|g_k + y_j\|^2} = \varepsilon_{k+1} + b_1 + k\pi$$

where $|\varepsilon_{k+1}| < \rho$. Take such an index j, and define

$$g_{k+1} := g_k + y_j$$

Again, standard continuity arguments show that there exist a positive number δ_{k+1} and an index n_{k+1} such that

$$\| \left(e^{\frac{1}{n_{k+1}}A} P_{\frac{h}{\|h\|}} \right)^{n_{k+1}} \frac{h}{\|h\|} - e^{(A\frac{g_{k+1}}{\|g_{k+1}\|}, \frac{g_{k+1}}{\|g_{k+1}\|})} \frac{g_{k+1}}{\|g_{k+1}\|} \| < 2\varepsilon$$

for all $h \in B(g_{k+1}, \delta_{k+1})$.
It is clear, by construction, that the sequence g_k converges. Let

$$h := \lim_{k \to \infty} g_k \quad \text{and} \quad g := \frac{h}{\|h\|}$$

Recall, that $\|g_1\| = 1$ and $\delta_1 < \frac{1}{2}$, therefore $\frac{1}{2} < \|g_k\| < \frac{3}{2}$ for all $k \geq 1$. It is also clear, by construction, that $h \in B(g_k, \delta_k)$ for all $k \geq 1$. Hence, for all $k \geq 1$ we have

$$\| \left(e^{\frac{1}{n_k}A} P_g \right)^{n_k} g - e^{(A\frac{g_k}{\|g_k\|}, \frac{g_k}{\|g_k\|})} \frac{g_k}{\|g_k\|} \| < 2\varepsilon$$

Notice, furthermore that

$$\begin{aligned}
\|e^{(A\frac{g_{2k+1}}{\|g_{2k+1}\|}, \frac{g_{2k+1}}{\|g_{2k+1}\|})} \frac{g_{2k+1}}{\|g_{2k+1}\|} - e^{(A\frac{g_{2k}}{\|g_{2k}\|}, \frac{g_{2k}}{\|g_{2k}\|})} \frac{g_{2k}}{\|g_{2k}\|}\| = \\
\|e^{a_{2k+1}} e^{(\varepsilon_{2k+1}+b_1+2k\pi)i} \frac{g_{2k+1}}{\|g_{2k+1}\|} - e^{a_{2k}} e^{(\varepsilon_{2k}+b_1+(2k-1)\pi)i} \frac{g_{2k}}{\|g_{2k}\|}\| \geq \\
\|e^{a_{2k+1}+b_1 i} g_1 - e^{a_{2k}+(b_1-\pi)i} g_1\| - \|e^{a_{2k+1}+b_1 i}(e^{\varepsilon_{2k+1}} \frac{g_{2k+1}}{\|g_{2k+1}\|} - g_1)\| - \\
-\|e^{a_{2k}+(b_1-\pi)i}(e^{\varepsilon_{2k}} \frac{g_{2k}}{\|g_{2k}\|} - g_1)\| \geq \\
2e^{a_1-1} - \|e^{a_{2k+1}+b_1 i}(e^{\varepsilon_{2k+1}} \frac{g_{2k+1}}{\|g_{2k+1}\|} - g_1)\| - \|e^{a_{2k}+(b_1-\pi)i}(e^{\varepsilon_{2k}} \frac{g_{2k}}{\|g_{2k}\|} - g_1)\|
\end{aligned}$$

We can now choose the values of $\varepsilon, \delta_1, \rho$ so small that

$$\|e^{a_{2k+1}+b_1 i}(e^{\varepsilon_{2k+1}} \frac{g_{2k+1}}{\|g_{2k+1}\|} - g_1)\| + \|e^{a_{2k}+(b_1-\pi)i}(e^{\varepsilon_{2k}} \frac{g_{2k}}{\|g_{2k}\|} - g_1)\| \leq e^{a_1-1}$$

and $5\varepsilon \leq e^{a_1-1}$
Then we have

$$\|(e^{\frac{1}{n_{2k+1}}A} P_g)^{n_{2k+1}} g - (e^{\frac{1}{n_{2k}}A} P_g)^{n_{2k}} g\| \geq \varepsilon$$

Therefore the sequence $(e^{\frac{1}{n}A} P_g)^n g$ does not converge, and the proof is complete.

□

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