Introduction to Algebra 1 **Problem sheet 3.** **2023.09.18-21.**

1. Show that
   a) the product of integers in the form $4k+1$ is again in the form $4k+1$,
   b) there exists infinitely many prime in the form $4k+3$.
   (Hint: what are the prime divisors of $4n!-1$)

2. Show that $\log_5 20$ is irrational! (Hint: use the Fundamental Theorem of Number Theory)

3. What is the exponent of 2 in the canonical representation of 17! (17 factorial)?

4. Show that $17^2 \nmid (n-5)(n+12)+51$ for all $n \in \mathbb{Z}$!

5. Show that if $(a^2, b) = (a, b^2)$, then $(a^3, b^7) = (a^7, b^3)$.

6. * What are those even numbers which have unique factorization in E (the nonunital ring of even numbers)?

7. * Consider the map $N : \mathbb{Z}[\sqrt{2}] \to \mathbb{Z}$, $N(a+b\sqrt{2}) = a^2 - 2b^2$. Show that
   a) $N(xy) = N(x)N(y)$ for all $x, y \in \mathbb{Z}[\sqrt{2}]$,
   b) $x \in \mathbb{Z}[\sqrt{2}]$ is a unit $\iff$ $|N(x)| = 1$,
   c) if $N(x) \in \mathbb{Z}$ is a prime number, then x is irreducible in $\mathbb{Z}[\sqrt{2}]$,
   d) $p \in \mathbb{Z}$ is a prime congruent to 3 or 5 modulo 8, then $p \in \mathbb{Z}[\sqrt{2}]$ is irreducible and
   e) 7, 17 and 23 are not irreducible in $\mathbb{Z}[\sqrt{2}]$!

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8. What is the greatest common divisor and least common multiple of the following integers?
   a) $2^{23}3^{10}7^{13}$ and $2^{15}7^{10}13^5$
   b) $2^{23}3^{10}7^{13}$, $2^{15}7^{10}13^5$ and $3^{15}7^{20}11^2$ (here first define the gcd and lcm of 3 integers!)

9. Show that any nonnegative integer can be written as a product of a square number and a squarefree number (n is squarefree if $p^2 \nmid n$ for all primes p, for example 6 and 30 are squarefree, but 12 and 18 are not squarefree as $2^2|12$ and $3^2|18$).

10. How many zero digits are at the end of the number 100! (100 factorial)?

11. Show, that for any a, b positive integers there exists m, n positive integers such that $m|a$, $n|b$, $(m, n) = 1$ and $mn = [a, b]$.

12. * For which primes p is the number $(2^p - 1)/p$ a square?

The problem sheets are available on the homepage of the lecturer: `www.math.bme.hu/~merdelyi/bevalg1/`