Introduction to Algebra 1 **Problem sheet 4.** **2023.09.25-28.**

1. What is 2^{67} modulo 61?
2. Solve the congruences (in $\mathbb{Z}$)
   a) $12x \equiv 15 \pmod{21}$,
   b) $12x \equiv 4 \pmod{6}$,
   c) $12x \equiv 4 \pmod{2}$ and
   d) $30x \equiv 4 \pmod{37}$.
3. Compute the table of the operations of $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/8\mathbb{Z}$! What is the table of multiplication in $(\mathbb{Z}/8\mathbb{Z})^*$?
4. Determine the following values:
   a) $\varphi(23)$, $\varphi(21)$, $\varphi(63)$ and $\varphi(10!)$,
   b) $120^{24} \mod 23$, $115^{21} \mod 21$, $68^{111} \mod 63$ and $111^{68} \mod 63$
   (be careful, 63 and 111 are not relatively prime!) and
   c) the last two digits of $3^{3^{3^4}}$.
5. * Prove Wilson's theorem: $(p-1)! \equiv -1 \pmod{p}$ for all prime numbers $p > 0$.
   Hints:
   (a) Make pairs of the form (a, a^{-1})! (why can this be done?)
   (b) Which pairs are not really pairs? (try it for some prime $p > 3$ and prove it using the fact that $\mathbb{Z}_p$ has no zero divisors as it is a domain.
   (c) Compute the product "pairwise".

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6. Solve the following system of congruences:

$$\begin{aligned} x &\equiv 2 \pmod{3} \\ x &\equiv 3 \pmod{8} \\ x &\equiv -4 \pmod{11} \end{aligned}$$

7. What is 3^{32} modulo 13?
8. a) What is $5^{-1} \pmod{26}$?
   b) Is 4 invertible modulo 26?
9. For which positive integers n is $\varphi(n) = 6$?
10. * Show that if H is a set, then $(\mathcal{P}(H), \cap, \Delta)$ is a domain. Here $\mathcal{P}(H)$ is the power set of H: the set of subsets of H, $\cap$ is the intersection and Δ is the symmetric difference: $A\Delta B = (A \cup B) - (A \cap B)$. Is $(\mathcal{P}(H), \cup, \cap)$ a ring?

The problem sheets are available on the homepage of the lecturer: `www.math.bme.hu/~merdelyi/bevalg1/`