Introduction to Algebra 1 **Problem sheet 5.** **2023.10.02-05.**

1. What is the algebraic form $(a+bi)$ of the following complex numbers?
   a) $(3-4i)(7+8i)$, b) $(3-4i)/(2-i)$, c) i^{2020} and d) $(1+i)^9$.

2. Solve the equation $z^2+2iz-1+i=0$ in $\mathbb{C}$!

3. Represent the solutions of the following equations on the plane!
   a) $|z-5+i|=2$, b) $|z-i|=|z+i|$, c) $|(z-3+4i)/(z-i)|\geq 1$,
   d) $|z|=3iz$, e) $|z|=iz$ and f) $z+\bar{z}<4$.

4. a) Give an explicit formula for $\binom{n}{0}-\binom{n}{2}+\binom{n}{4}-\dots$ by comparing the algebraic and trigonometric form of $(1+i)^n$!

   b) Compute $(\cos x+i\sin x)^3$ in two different ways! With the help of this express $\cos(3x)$ as a funciton of $\cos x$!

5. Let z_1, z_2 and $z_3\in\mathbb{C}$ and $w=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$. Prove that the following are equivalent:
   1) $z_1z_2z_3$ is an equilateral triangle with vertices in counterclockwise order and
   2) $z_1+z_2w+z_3w^2=0$.

6. What is the sum and the product of primitive 5th and 8th roots of unity?

7. a) Show that the units of $\mathbb{Z}/m\mathbb{Z}$ are exactly the reduced residue classes!
   b) Show that the following are equivalent:

   1. m is a prime 2. $\mathbb{Z}/m\mathbb{Z}$ is a domain 3.$\mathbb{Z}/m\mathbb{Z}$ is a field

8. Let $\mathbb{H}=\{a+bi+cj+dk|a,b,c,d\in\mathbb{R}\}$ be a set and define the following operations:
   $+$ in the obvious way (i. e. $(a+bi+cj+dk)+(a'+b'i+c'j+d'k)=(a+a)'+(b+b')i+(c+c')j+(d+d')k)$
   $\cdot$ by extending the relations $i^2=j^2=k^2=-1$, $ij=-ji=k$, $jk=-kj=i$ and $ki=-ik=j$ distributively (for example $(j+2k)(j-3i)=j^2-3ji+2kj-6ki=-1+3k-2i-6j$)
   a) Show that $(\mathbb{H},+,\cdot)$ is a non-commutative ring! (It is called the ring of Hamilton quaternions)
   b) Show that any nonzero element has an inverse! (Thus $(\mathbb{H},+,\cdot)$ is a skew field).
   Hint: $(a+bi+cj+dk)(a-bi-cj-dk)\in\mathbb{R}$ for any $a,b,c,d\in\mathbb{R}$.

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9. Let $z=1+3i$ and $w=2-i$. Compute
   a) $z\bar{z}$, b) $w/\bar{w}$, c) $|z-w|$, d) $|2z-zw|$ and e) $|w/z\bar{w}^3|$.

10. What are the square roots of the complex number $1-2i$? (i. e. the numbers $w=x+yi$ such that $w^2=1-2i$)

11. Let ε be a primitive n-th root of unitiy. What are the possible orders of a) $-\varepsilon$ and b) ε^k?

12. What are the fifth roots of $-\sqrt{3}+i$?

13. 2 and i are two vertices of a square. What can be the other vertices?

14. Prove that in any ring R the identity $r\cdot 0=0=0\cdot r$ holds for any $r\in R$.

The problem sheets are available on the homepage of the lecturer: `www.math.bme.hu/~merdelyi/bevalg1/`