You have 120 minutes to work, and you need to get at least 40 points (of 80).

In the following problems no explanation is needed, just mark or write the correct answer(s). The first exercise is worth $6 = 6 \cdot 1$ points, the others 2-2 points.

A1 Which of the following statements are true?

(a) $\lfloor -x \rfloor = -\lceil x \rceil$ for all $x \in \mathbb{R}$.

(b) If $\{a_1, a_2, \ldots, a_m\}$ is a complete residue system modulo m and $b \neq 0$ is an integer, then $\{ba_1, ba_2, \ldots, ba_m\}$ is also a complete residue system.

(c) If $p \in \mathbb{F}[x]$ is irreducible, then it has no roots in $\mathbb{F}$.

(d) Any two distinct nonzero vectors are linearly independent over $\mathbb{Z}_2$.

(e) If $A \in \mathbb{F}^{n\times n}$ is invertible, then for all i there exists j such that the cofactor $A_{ij} \neq 0$.

(f) If $A \in \mathbb{F}^{n\times n}$ is of full rank, then $A^T \cdot A$ is invertible.

A2 Which of the following subsets of $\mathbb{R}$ are well-ordered?
The interval $A = [0, 1]$, $B = \left\{ \frac{1}{n} \middle| n \in \mathbb{N} \right\}$ and $C = \left\{ 1 - \frac{1}{n} \middle| n \in \mathbb{N} \right\}$

A3 Convert 2020 to base 8!

A4 Solve the congruence $5x \equiv 26 \pmod{23}$.

A5 Enumerate the fourth roots of $(1 + 2i)^4$ in algebraic form!

A6 Give an example for a prime p such that $x^3 + 2x + 5$ is irreducible mod p.

A7 Write $x^9 - 1$ as a product of cyclotomic polynomials. What is the degree of the terms?

A8 What is the dimension of the nullspace of $\begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 3 & 2 & 3 \end{pmatrix}$ viewed over $\mathbb{Q}$ and $\mathbb{Z}_2$?

A9 Give an example of nonzero real matrices A and B such that $\mathrm{rk}(A + B) = \mathrm{rk}(A) + \mathrm{rk}(B)$!

A10 If $\det(A) = 2$ for $A = (\underline{a}, \underline{b}, \underline{c})$, what is $\det(B)$ for $B = (2\underline{c} - \underline{a}, -\underline{c}, 3\underline{b})$? (the vectors are the column vectors of the matrices)

For the following problems write detailed explanation! Each problem is worth 8 points.

B1 Solve the following system of congruences:

$$\begin{aligned} x &\equiv 2 \pmod 4 \\ x &\equiv -1 \pmod 3 \\ x &\equiv 3 \pmod 7 \end{aligned}$$

B2 Solve the equation $\dfrac{z^3}{z^3+i} = 1+i$ over $\mathbb{C}$!

B3 Decompose the polynomial $f(x) = x^5 + x^4 - 2x^2 + 2$ as a product of irreducibles over $\mathbb{Q}$ and $\mathbb{Z}_5$!

B4 Let a, b and c be the complex roots of the polynomial $2x^3 - 2x^2 + 3$. Determine the value of $a^2b^2 + b^2c^2 + c^2a^2$!

B5 Consider the following linear map $\mathbb{R}^3 \to \mathbb{R}^3$:
φ : the reflection to the plane yz
ψ : the rotation by the axis z with $90°$.

What are the standard matrices of φ, ψ and $\psi \circ \varphi$? In the composition the reflection is done first and then the rotation.

Compute $\psi(\varphi((1,2,3)^T))$ as well!

B6 Determine the rank of the following matrix A. Give a basis of $\mathcal{C}(A)$ and $\mathcal{N}(A)$. Give a nonzero subdeterminant of size $\mathrm{rk}(A)$!

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & -1 & 0 \\ 1 & 3 & 2 & 1 \end{pmatrix}$$

B7 Compute the determinant

$$\left| \binom{j+k-2}{j-1} \right|_{n\times n} = \begin{vmatrix} \binom{0}{0} & \binom{1}{1} & \cdots & \binom{n-1}{n-1} \\ \binom{1}{0} & \binom{2}{1} & \cdots & \binom{n}{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n-1}{0} & \binom{n}{1} & \cdots & \binom{2n-2}{n-1} \end{vmatrix}.$$