

Probability 1
CEU Budapest, fall semester 2013
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Homework sheet 1 – due on 23.09.2013

1. Define a σ -algebra as follows:

Definition 1 For a nonempty set Ω , a family \mathcal{F} of subsets of ω (i.e. $\mathcal{F} \subset 2^\Omega$, where $2^\Omega := \{A : A \subset \Omega\}$ is the power set of Ω) is called a σ -algebra over Ω if

- $\emptyset \in \mathcal{F}$
- if $A \in \mathcal{F}$, then $A^C := \Omega \setminus A \in \mathcal{F}$ (that is, \mathcal{F} is closed under complement taking)
- if $A_1, A_2, \dots \in \mathcal{F}$, then $(\cup_{i=1}^\infty A_i) \in \mathcal{F}$ (that is, \mathcal{F} is closed under countable union).

Show from this definition that a σ -algebra is closed under countable intersection, and under finite union and intersection.

2. (a) We toss a biased coin, on which the probability of heads is some $0 \leq p \leq 1$. Define the random variable ξ as the indicator function of tossing heads, that is

$$\xi := \begin{cases} 0, & \text{if tails} \\ 1, & \text{if heads} \end{cases} .$$

- i. Describe the distribution of ξ (called the Bernoulli distribution with parameter p) in the “classical” way, listing possible values and their probabilities,
- ii. and also by describing the distribution as a measure on \mathbb{R} , giving the weight $\mathbb{P}(\xi \in B)$ of every Borel subset B of \mathbb{R} .
- iii. Calculate the expectation of ξ .

(b) We toss the previous biased coin n times, and denote by X the number of heads tossed.

- i. Describe the distribution of X (called the Binomial distribution with parameters (n, p)) by listing possible values and their probabilities.
- ii. Calculate the expectation of X by integration (actually summation in this case) using its distribution,
- iii. and also by noticing that $X = \xi_1 + \xi_2 + \dots + \xi_n$, where ξ_i is the indicator of the i -th toss being heads, and using linearity of the expectation.

3. The Fatou lemma is the following

Theorem 1 Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and f_1, f_2, \dots a sequence of measurable functions $f_n : \Omega \rightarrow \mathbb{R}$, which are nonnegative, e.g. $f_n(x) \geq 0$ for every $n = 1, 2, \dots$ and every $x \in \Omega$. Then

$$\int_{\Omega} \liminf_{n \rightarrow \infty} f_n(x) \, d\mu(x) \leq \liminf_{n \rightarrow \infty} \int_{\Omega} f_n(x) \, d\mu(x)$$

(and both sides make sense).

Show that the inequality in the opposite direction is in general false, by choosing $\Omega = \mathbb{R}$, μ as the Lebesgue measure on \mathbb{R} , and constructing a sequence of nonnegative $f_n : \mathbb{R} \rightarrow \mathbb{R}$ for which $f_n(x) \xrightarrow{n \rightarrow \infty} 0$ for every $x \in \mathbb{R}$, but $\int_{\mathbb{R}} f_n(x) \, dx \geq 1$ for all n .

4. The *ternary* number $0.a_1a_2a_3\dots$ is the analogue of the usual decimal fraction, but writing numbers in base 3. That is, for any sequence a_1, a_2, a_3, \dots with $a_n \in \{0, 1, 2\}$, by definition

$$0.a_1a_2a_3\dots := \sum_{n=1}^{\infty} \frac{a_n}{3^n}.$$

Now let us construct the ternary fraction form of a random real number X via a sequence of fair coin tosses, such that we rule out the digit 1. That is,

$$a_n := \begin{cases} 0, & \text{if the } n\text{-th toss is tails,} \\ 2, & \text{if the } n\text{-th toss is heads} \end{cases},$$

and setting $X = 0.a_1a_2a_3\dots$ (ternary). In this way, X is a “uniformly” chosen random point of the famous *middle-third Cantor set* C defined as

$$C := \left\{ \sum_{n=1}^{\infty} \frac{a_n}{3^n}, a_n \in \{0, 2\} (n = 1, 2, \dots) \right\}.$$

Show that

- (a) The distribution of X gives zero weight to every point – that is, $\mathbb{P}(X = x) = 0$ for every $x \in \mathbb{R}$. (As a consequence, the cumulative distribution function of X is continuous.)
- (b) The distribution of X is not absolutely continuous w.r.t the Lebesgue measure on \mathbb{R} .