Probability I at CEU, Final exam, fall 2013 $\frac{1}{3}$ Flogy

① a.) Let $f_t(\omega) := e^{itX(\omega)}$ for $t \in \mathbb{R}$, $\omega \in \Omega$. Then, as $t \to t_0$, $f_t(\omega) \to f_{t_0}(\omega)$ for every $\omega \in \Omega$. Moreover $|f_t(\omega)| \le g(\omega) \equiv 1$ for $\forall \omega \in \Omega$ and $g : \Omega \to \mathbb{R}$ is integrable, so the dominated convergence theorem gives

$$\varphi(t) = \int_\Omega f_t(\omega)\, d\mathbb{P}(\omega) \xrightarrow{t \to t_0} \int_\Omega f_{t_0}(\omega)\, d\mathbb{P}(\omega) = \varphi(t_0) \quad \square$$

b.) ~~Let f_t~~ Fix $t_0 \in \mathbb{R}$ and let $f_h(\omega) = \dfrac{e^{i(t_0+h)X(\omega)} - e^{itX(\omega)}}{h}$.

Since $t \to e^{it}$ is Lipschitz cont. with constant 1 (meaning $|e^{i(t+\Delta)} - e^{it}| \le \Delta$), we have that

$$|f_h(\omega)| \le |X(\omega)| =: g(\omega) \quad \text{for } \forall \omega \in \Omega$$

By assumption this g is integrable.

Moreover, $\lim\limits_{h \to 0} f_h(\omega) = \dfrac{d}{dt} e^{itX(\omega)} = iX(\omega) e^{itX(\omega)}$ for $\forall \omega \in \Omega$,

so the dominated convergence theorem gives that

$$\varphi'(t) = \lim_{h \to 0} \frac{\varphi(t+h) - \varphi(t)}{h} = \lim_{h \to 0} \int_\Omega f_h(\omega)\, d\mathbb{P}(\omega) = \int_\Omega iX(\omega) e^{itX(\omega)}\, d\mathbb{P}(\omega)$$

(and this exists $\in \mathbb{C}$).

For $t = 0$, $\varphi'(0) = \int_\Omega iX\, d\mathbb{P}(\omega) = i\mathbb{E}X$. $\square$

② $I_n := \int_0^1 \int_0^1 \cdots \int_0^1 f\left(\frac{x_1 + \dots + x_n}{n}\right) dx_1 \dots dx_n = \mathbb{E}\left(f\left(\frac{X_1 + \dots + X_n}{n}\right)\right)$, where $X_1, X_2, \dots, X_n$ are i.i.d $\sim \mathrm{Uni}[0,1]$

Since $f : [0,1]$ is continuous it is also automatically bounded, so the weak law of large numbers, which says ~~exactly~~ that says in particular that

$$\frac{X_1 + \dots + X_n}{n} \Rightarrow \mathbb{E}\,\mathrm{Uni}[0,1] = \frac{1}{2}$$

$$\underline{I_n \to f\left(\tfrac{1}{2}\right).} \quad \square$$

(3) $\mathbb{E}X_i = P(X_i=1) = \frac{1}{4} =: m$ for every i. If $|i-j|>1$, then X_i and X_j are independent, so $\mathrm{Cov}(X_i,X_j)=0$.

Moreover, the X_i are uniformly distributed with finite variance, so $\mathrm{Var}\,X_i \le C_1$ and $\mathrm{Cov}(X_i, X_{i+1}) \le C_2$. These imply that $\mathbb{E}\,S_n = n\cdot m$

$$\mathrm{Var}\,S_n = \mathrm{Cov}(S_n,S_n) = \sum_{i,j=1}^{n}\mathrm{Cov}(X_i,X_j) \le nC_1 + 2(n-1)C_2 \le C\cdot n$$

~~so Var $\frac{S_n}{n}$~~ so $\mathbb{E}\,\frac{S_n}{n} = m$ and $\mathrm{Var}\,\frac{S_n}{n} \le \frac{C}{n} \to 0$.

So the standard argument gives that $\boxed{\frac{S_n}{n} \Rightarrow m}$: for $\forall\,\varepsilon>0$

$$P\left(\left|\frac{S_n}{n}-m\right|>\varepsilon\right) \overset{\text{Chebys.}}{\le} \frac{\mathrm{Var}\,\frac{S_n}{n}}{\varepsilon^2} \xrightarrow{n\to\infty} 0$$

□

Summary: $\frac{S_n}{n} \to \frac{1}{4}$ weakly.

(4) Let $\xi_{n,k}$ be i.i.d. $\sim X$ for $n,k = 0,1,2,\dots$ where

k	0	1	2	3
$P(X=k)$	1/4	1/4	1/4	1/4

Then $Z_0 = 1$ and $Z_{n+1} = \sum_{k=1}^{Z_n} \xi_{nk}$ by the definition of the branching process.

$\mu := \mathbb{E}X = \frac{0+1+2+3}{4} = \frac{3}{2}$, so an easy calculation shows that $\frac{Z_n}{\mu^n}$ is a martingale. Since $\frac{Z_n}{\mu^n} \ge 0$ the martingale convergence theorem immediately gives that the limit $\Theta := \lim_{n\to\infty}\frac{Z_n}{\mu^n}$ exists almost surely.

To see that the limit is not identically zero, we need to calculate $\mathrm{Var}\,\frac{Z_n}{\mu^n}$ and see that it converges to some finite limit. In particular, $\sup_n \mathrm{Var}\,\frac{Z_n}{\mu^n} < \infty$, so the $\underline{L^2}$ martingale convergence theorem ensures that $\frac{Z_n}{\mu^n} \to \Theta$ also in L^2, implying that $1 = \mathbb{E}\,\frac{Z_n}{\mu^n} \to \mathbb{E}\Theta$, so $\mathbb{E}\Theta = 1$ meaning $\Theta \not\equiv 0$. □

(5) Let $\mathcal{F}_n = \sigma(X_1 \dots X_n)$. and $Y_n := 2^{X_n}$

$$\mathbb{E}(Y_{n+1} \mid \mathcal{F}_n) = \begin{cases} \frac{2}{3} 2^{X_n - 1} + \frac{1}{3} 2^{X_n + 1} = 2^{X_n} & \text{on } X_n \notin \{-10; 10\} \\ 1 \cdot 2^{X_n} = 2^{X_n} & \text{on } X_n \in \{-10; 10\} \end{cases} \left(= Y_n, \right.$$

So Y_n is indeed a martingale.

Y_n is bounded, so the martingale convergence theorem ensures that it is a.s. convergent. Of course, the limit has to be 2^{-10} or 2^{10} (everywhere else the flea keeps jumping distance 1 at each step). So an endpoint has to be reached a.s.

Let T be the stopping time $T := \min\{n : X_n \in \{-10; 10\}\}$.

Then we have seen that $\mathbb{P}(T < \infty) = 1$, and since Y_n is bounded, the optional stopping theorem gives $\mathbb{E}Y_T = \mathbb{E}Y_0 = 2^0 = 1$.

But
$$\begin{cases} \mathbb{E} Y_T = \mathbb{P}(X_\infty = -10) \cdot 2^{-10} + \mathbb{P}(X_\infty = +10) \cdot 2^{10} = 1 \\ \mathbb{P}(X_\infty = -10) + \mathbb{P}(X_\infty = 10) = 1 \end{cases}$$

Solving this system of equations gives

$$\mathbb{P}(X_\infty = -10) = \frac{1}{1 + 2^{-10}} = \frac{1024}{1025} \quad \text{and} \quad \boxed{\mathbb{P}(X_\infty = 10) = \frac{2^{-10}}{1+2^{-10}}}$$

$$\boxed{\mathbb{P}(X_\infty = 10) = \frac{1}{1025}}$$