Probability 1
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Homework sheet 3 – solutions

3.1 **(homework)** *Poisson approximation of the binomial distribution.* Fix $0 < \lambda \in \mathbb{R}$. Show that if X_n has binomial distribution with parameters (n, p) such that $np \to \lambda$ as $n \to \infty$, then X_n converges to $Poi(\lambda)$ weakly.

Solution: Set $q_n = 1 - p_n$, so X_n has characteristic function

$$\psi_{X_n}(t) = \left(q_n + p_n e^{it}\right)^n = \left[\left(1 + \frac{e^{it} - 1}{1/p_n}\right)^{1/p_n}\right]^{np_n}.$$

The base of the power converges to $\exp(e^{it} - 1)$ as $p_n \to 0$ by standard elementary calculus, while the exponent converges to λ, so

$$\psi_{X_n}(t) \to e^{\lambda(e^{it}-1)},$$

which is exactly the characteristic function of the $Poi(\lambda)$ distribution, so the continuity theorem ensures that X_n converges to $Poi(\lambda)$ weakly.

3.2 **(homework)** Let X be uniformly distributed on $[-1; 1]$, and set $Y_n = nX$.

a.) Calculate the characteristic function ψ_n of Y_n.

b.) Calculate the pointwise limit $\lim_{n\to\infty} \psi_n(t)$, if it exists.

c.) Does (the distribution of) Y_n have a weak limit?

d.) How come?

Solution:

a.) The characteristic function of X is

$$\psi_1(t) = \int_0^1 e^{itx} \frac{1}{2}\, \mathrm{d}x = \frac{1}{2}\left[\frac{e^{itx}}{it}\right]_0^1 = \frac{\sin t}{t},$$

so

$$\psi_n(t) = \psi_1(nt) = \frac{\sin(nt)}{nt}$$

(with $\psi_n(0) = 1$, of course).

b.) So for every $t \neq 0$ we have $|\psi_n(t)| \leq \frac{1}{n|t|}$, which goes to 0 as $n \to \infty$, so

$$\lim_{n\to\infty} \psi_n(t) = \begin{cases} 0, & \text{if } t \neq 0 \\ 1, & \text{if } t = 0. \end{cases}$$

c.) No: $\mathbb{P}(Y_n < x) \to \frac{1}{2}$ as $n \to \infty$ for every $x \in \mathbb{R}$, and the constant $\frac{1}{2}$ is not a distribution function. Another possible reasoning is that for any continuous $\varphi : \mathbb{R} \to \mathbb{R}$ which is bounded by some K and supported on some bounded interval $[a, b]$ we have

$$|\mathbb{E}\varphi(Y_n)| \leq \mathbb{E}|\varphi(Y_n)| \leq K\mathbb{P}(Y_n \in [a, b]) \leq K\frac{b-a}{2n} \xrightarrow{n\to\infty} 0,$$

so if Y_n would converge weakly to some Y, then we would have $\mathbb{E}\varphi(Y) = 0$ for every such φ, but then the distribution of Y has to give zero weight to every interval, which is impossible.

d.) There is no contradiction with the continuity theorem, because the pointwise limit $\psi(t) := \lim_{n\to\infty} \psi_n(t)$ of the sequence of characteristic functions is not continuous at 0 (and thus not a characteristic function).

3.3 Let $X_1, X_2, \dots$ be independent random variables such that

$$\mathbb{P}(X_n = n^2 - 1) = \frac{1}{n^2}, \quad \mathbb{P}(X_n = -1) = 1 - \frac{1}{n^2}.$$

Show that $\mathbb{E}X_n = 0$ for every n, but

$$\lim_{n\to\infty} \frac{X_1 + \dots X_n}{n} = -1$$

almost surely.

3.4 *Exchangeability of integral and limit.* Consider the sequences of functions $f_n : [0,1] \to \mathbb{R}$ and $g_n : [0,1] \to \mathbb{R}$ concerning their pointwise limits and the limits of their integrals. Do there exist integrable functions $f : [0,1] \to \mathbb{R}$ and $g : [0,1] \to \mathbb{R}$, such that $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$ for Lebesgue almost every $x \in [0,1]$? What is $\lim_{n\to\infty} \left(\int_0^1 f_n(x)dx \right)$ and $\lim_{n\to\infty} \left(\int_0^1 g_n(x)dx \right)$? Are the conditions of the dominated and monotone convergence theorems and the Fatou lemma satisfied? If yes, what do these theorems ensure about these specific examples?

(a)

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \le x < 1/n, \\ 2n - n^2 x & \text{if } 1/n \le x \le 2/n, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Write n as $n = 2^k + l$, where $k = 0, 1, 2\dots$ and $l = 0, 1, \dots, 2^k - 1$ (this can be done in a unique way for every n). Now let

$$g_n(x) = \begin{cases} 1 & \text{if } \frac{l}{2^k} \le x < \frac{l+1}{2^k}, \\ 0 & \text{otherwise.} \end{cases}$$

3.5 **(homework)** *Exchangeability of integrals.* Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 < x,\ 0 < y \text{ and } 0 \le x - y \le 1, \\ -1 & \text{if} \quad 0 < x,\ 0 < y \text{ and } 0 < y - x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(x,y)dx \right) dy$ and $\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(x,y)dy \right) dx$. What's the situation with the Fubini theorem?

Solution: Sketching the function one easily sees that

$$\int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}x = \begin{cases} 1 - y, & \text{if } 0 < y < 1 \\ 0, & \text{if not} \end{cases},$$

so $\int\limits_{-\infty}^{+\infty} \left(\int\limits_{-\infty}^{+\infty} f(x,y)dx \right) dy = \int\limits_{0}^{1} (1-y)dy = \frac{1}{2}$. Similarly

$$\int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}y = \begin{cases} -1+x, & \text{if } 0 < x < 1 \\ 0, & \text{if not} \end{cases} \quad ,$$

so $\int\limits_{-\infty}^{+\infty} \left(\int\limits_{-\infty}^{+\infty} f(x,y)dy \right) dx = \int\limits_{0}^{1} (x-1)dx = -\frac{1}{2}$. The two double integrals are not equal, but this does not contradict the Fubini theorem, because f is not integrable (w.r.t. Lebesgue measure on $\mathbb{R}^2$). Indeed, $\iint_{\mathbb{R}^2} |f| = \infty$.

3.6 *Weak convergence and densities.*

(a) **(homework)** Prove the following

Theorem 1 *Let $\mu_1, \mu_2, \ldots$ and μ be a sequence of probability distributions on $\mathbb{R}$ which are absolutely continouos w.r.t. Lebesgue measure. Denote their densities by $f_1, f_2, \ldots$ and f, respectively. Suppose that $f_n(x) \overset{n\to\infty}{\longrightarrow} f(x)$ for every $x \in \mathbb{R}$. Then $\mu_n \Rightarrow \mu$ (weakly).*

(Hint: denote the cumulative distribution functions by $F_1, F_2, \ldots$ and F, respectively. Use the Fatou lemma to show that $F(x) \le \liminf_{n\to\infty} F_n(x)$. For the other direction, consider $G(x) := 1 - F(x)$.

Solution: $F_n(x) = \int_{-\infty}^{x} f_n(x)\,\mathrm{d}x$ and $f_n(x) \to f(x)$ for every x, so the Fatou lemma says that

$$F(x) = \int_{-\infty}^{x} f(x)\,\mathrm{d}x = \int_{-\infty}^{x} \liminf_{n\to\infty} f_n(x)\,\mathrm{d}x \le \liminf_{n\to\infty} \int_{-\infty}^{x} f_n(x)\,\mathrm{d}x = \liminf_{n\to\infty} F_n(x).$$

Similarly,

$$\begin{aligned} 1 - F(x) &= \int_{x}^{\infty} f(x)\,\mathrm{d}x = \int_{x}^{\infty} \liminf_{n\to\infty} f_n(x)\,\mathrm{d}x \\ &\le \liminf_{n\to\infty} \int_{x}^{\infty} f_n(x)\,\mathrm{d}x = \liminf_{n\to\infty} (1 - F_n(x)) = 1 - \limsup_{n\to\infty} F_n(x), \end{aligned}$$

which implies $\limsup_{n\to\infty} F_n(x) \le F(x)$, so $F_n(x) \to F(x)$ for every x, and we are done.

(b) Show examples of the following facts:

i. It can happen that the f_n converge pointwise to some f, but the sequence μ_n is not weakly convergent, because f is not a density.

ii. It can happen that the μ_n are absolutely continuous, $\mu_n \Rightarrow \mu$, but μ is not absolutely continuous.

iii. It can happen that the μ_n and also μ are absolutely continuous, $\mu_n \Rightarrow \mu$, but $f_n(x)$ does not converge to $f(x)$ for any x.