$\boxed{7.1}$ (Durrett 5.1.4)

a.) Existence: $Y := \lim_{M\to\infty} Y_M$ exists, because Y_n is increasing in M.

The monotone convergence theorem implies that for $A \in \mathcal{F}$

$$\int_A Y\,dP \overset{Y_M \nearrow Y}{=} \lim_{M\to\infty} \int_A Y_M\,dP \xlongequal{Y_n = \mathbb{E}(X_M|\mathcal{F})} \lim_{M\to\infty} \int_A X_M\,dP \xlongequal{X_M \nearrow X} \int_A X\,dP,$$

so Y will do.

b.) Uniqueness If Y and Z are both $\mathcal{F}$-measurable and $\int_A Y\,dP = \int_A Z\,dP$ for every $A \in \mathcal{F}$, then $Y = Z$ a.s. because for $\forall \varepsilon > 0, M < \infty$

$A_{\varepsilon,M} := \{\omega \mid Y + \varepsilon \le Z \le M\}$ is $\mathcal{F}$-measurable,

so $0 = \int_A (Z - Y)\,dP \ge \varepsilon\,\mathbb{P}(A_{\varepsilon,M}) \Rightarrow \mathbb{P}(A_{\varepsilon,M}) = 0 \quad \forall \varepsilon, M$

but $\{Z > Y\} = \bigcup_n A_{\frac{1}{n}, n}$, so $\mathbb{P}(Z > Y) = 0$

and similarly $\mathbb{P}(Y > Z) = 0$ □

$\boxed{7.6}$ (Durrett 5.1.10)

Let $g(z) := \sum_{k=0}^{\infty} \mathbb{P}(N=k) z^k = \mathbb{E}(z^N)$ be the generating fn. of N.

Let $\Psi_Y(t) = \mathbb{E}(e^{itY_k})$ be the common characteristic fn. of the Y_k

and let $\Psi_X(t) = \mathbb{E}(e^{itX})$ be the char. fn of X.

Then $g(1) = 1$, $g'(1) = \mathbb{E}N$ and $g''(1) = \mathbb{E}(N^2 - N)$

$\Psi_Y(0) = 1$, $\Psi_Y'(0) = i\mu$ and $\Psi_Y''(0) = -(\sigma^2 + \mu^2)$

$\Psi_X'(0) = i\mathbb{E}X$ and $\Psi_X''(0) = -(\operatorname{Var}X + \mathbb{E}^2 X)$

The theorem of total expectation gives

$$\Psi_X(t) = \sum_{k=0}^{\infty} \mathbb{P}(N=k)\,\mathbb{E}(e^{itX} \mid N=k) = \sum_{k=0}^{\infty} \mathbb{P}(N=k)\left(\mathbb{E}e^{itY_1}\right)^k = g(\Psi_Y(t)).$$

Differentiating this, we get

$$\psi_X'(t) = g'(\psi_Y(t))\,\psi_Y'(t) \xrightarrow{t=0} \psi_X'(0) = g'(1)\cdot\psi_Y'(0)$$

$$i\mathbb{E}X = \mathbb{E}N \cdot i\mu$$

$$\psi_X''(t) = g''(\psi_Y(t))\left[\psi_Y'(t)\right]^2 + g'(\psi_Y(t))\,\psi_Y''(t) \qquad \mathbb{E}X = \mu\,\mathbb{E}N$$

$$\Downarrow t=0$$

$$\psi_X''(0) = g''(1)\left[\psi_Y'(0)\right]^2 + g'(1)\,\psi_Y''(t)$$

$$-E^2X - \operatorname{Var}X = (\mathbb{E}N^2 - \mathbb{E}N)(i\mu)^2 + \mathbb{E}N(-\sigma^2-\mu^2)$$

$$\operatorname{Var}X = \mu^2 EN^2 - \cancel{\mu^2 EN} + \sigma^2\mathbb{E}N + \cancel{\mu^2 EN} - \mu^2E^2N = \sigma^2 EN + \mu^2\operatorname{Var}N \quad \square$$

$\boxed{4.9}$ (Durrett 5.2.3) $X_n = -\frac{1}{n}$ is increasing, so it is a submartingale,

but $X_n^2 = \frac{1}{n^2}$ is decreasing, so it is a supermartingale.

$\boxed{4.10}$ (Durrett 5.2.4) Let $\mathbb{P}(\xi_k = -1) = 1 - 2^{-k} = 1 - \mathbb{P}(\xi_k = 2^k - 1)$,

so $\mathbb{E}\xi_k = (1-2^{-k})(-1) + 2^{-k}(2^k-1) = -1 + 2^{-k} + 1 - 2^{-k} = 0$, but the Borel–Cantelli lemma says that with probability 1,

$\xi_k = -1$ for all but finitely many k-s.

So $X_n = \xi_1 + \dots + \xi_n$ is a martingale, but almost surely $X_n \to -\infty$. $\square$