Tools of Modern Probability

Imre Péter Tóth

Exercise sheet 4, fall 2018

4.1 Let $(X, \mathcal{F})$ be a measurable space and let μ, ν be σ-finite measures on it. Show that there is a countable partition $X = \dot{\bigcup}_i A_i$ such that $\mu(A_i) < \infty$ and $\nu(A_i) < \infty$ for every i. Use this to show that the special case of the Radon-Nikodym theorem for finite measures implies the general theorem (for σ-finite measures).

4.2 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $X : \Omega \to \mathbb{R}^+$ be integrable and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. Define $\nu : \mathcal{G} \to \mathbb{R}^+$ by $\nu(A) := \int_A X \,\mathrm{d}\mathbb{P}$ (whenever $A \in \mathcal{G}$). Check that ν is a measure on $(\Omega, \mathcal{G})$.

4.3 Let X be a nonempty set and let $\mathcal{F}_i \subset 2^X$ be a σ-algebra for every $i \in I$, where I is some index set. I may be arbitrary (possibly much bigger that countable), but we assume $I \neq \emptyset$. Show that $\mathcal{F} := \bigcap_{i\in I} \mathcal{F}_i$ is also a σ-algebra. (Note that the assumption $I \neq \emptyset$ is important.)

4.4 Let $(\Omega, \mathcal{F})$ be a probability space and let $X : \Omega \to \mathbb{R}$ be (Borel-)measurable. Let $(\mathcal{G}_i)_{i\in I}$ be the family of all σ-algebras over Ω such that X is $\mathcal{G}_i$-measurable, and let $\mathcal{G} := \bigcap_{i\in I} \mathcal{G}_i$. Show that $\mathcal{G}$ is the *smallest* σ-algebra for which X is measurable. (In what sense exactly is it the smallest?)

4.5 Let $(\Omega, \mathcal{F})$ be a probability space, let $X : \Omega \to \mathbb{R}$ be $(\mathcal{F}, \mathcal{B})$-measurable, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$. Let $\sigma(X)$ be the smallest σ-algebra on Ω for which X is measurable. (This exists by the previous exercise.) This is called the *σ-algebra generated by X*. Show that

$$\sigma(X) = \{X^{-1}(B) \,|\, B \in \mathcal{B}\}.$$

4.6 Let $(\Omega, \mathcal{F})$ be a probability space, and let $\mathcal{G}_1, \mathcal{G}_2 \subset \mathcal{F}$ be sub-σ-algebras. We say that $\mathcal{F}_1$ and $\mathcal{F}_1$ are independent if any $A \in \mathcal{G}_1$ and $B \in \mathcal{G}_2$ are independent. Show that if the random variables X and Y are independent, then $\sigma(X)$ and $\sigma(Y)$ are independent.

4.7 Let $(\Omega, \mathcal{F})$ be a probability space, and let $\mathcal{G}_1, \mathcal{G}_2 \subset \mathcal{F}$ be sub-σ-algebras. Let X and Y be random variables, $X \in \mathcal{G}_1$, $Y \in \mathcal{G}_2$. Show that if $\sigma(X)$ and $\sigma(Y)$ are independent, then X and Y are independent.

4.8 Show that if X is a random variable, $f : \mathbb{R} \to \mathbb{R}$ measurable and $Y = f(X)$, then $\sigma(Y) \subset \sigma(X)$. Show an example when equality holds, and an example when not.

4.9 Show that if X, Y are independent random variables and $f, g : \mathbb{R} \to \mathbb{R}$ are measurable, then $f(X)$ and $g(Y)$ are also independent.

4.10 Show that the random variables $X, Y : \Omega \to \mathbb{R}$ are independent if and only if the (joint) distribution of the pair (X, Y) (which is a probability measure on $\mathbb{R}^2$) is the product of the distributions of X and Y.

4.11 Show that if X and Y are independent and integrable, then $\mathbb{E}(XY) = \mathbb{E}X\mathbb{E}Y$.

4.12 Show that if the random variable X is independent of the σ-algebra $\mathcal{G}$, then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}X$.

4.13 Let $\Omega = \{a, b, c\}$ and $\mathbb{P}$ the uniform measure on it. Let $X = \mathbf{1}_{\{c\}}$ and let $\mathcal{G} = \{\emptyset, \{a\}, \{b, c\}, \Omega\}$. Calculate $\mathbb{E}(X|\mathcal{G})$.

4.14 We roll two fair dice and let X, Y be the numbers rolled. Calculate $\mathbb{E}(X|X + Y)$.

4.15 Let $\Omega = [0, 1]^2$ and let $\mathbb{P}$ be Lebesgue measure on Ω. Let $X, Y : \Omega \to \mathbb{R}$ be defined as $X(u, v) = u$ and $Y(u, v) = \sqrt{u + v}$. Calculate $\mathbb{E}(Y|X)$.

4.16 Let U and V be independent random variables, uniformly distributed on $[0, 1]$. Calculate $\mathbb{E}(\sqrt{U+V}|U)$.

4.17 Let U and V be independent random variables, uniformly distributed on $[0, 1]$. Calculate $\mathbb{E}(U+V|U-V)$.

4.18 Let U and V be independent random variables, uniformly distributed on $[0, 1]$. Calculate $\mathbb{E}(\sqrt{U+V}|U-V)$.

4.19 Let X and Y be independent standard Gaussian random variables. Let $U = X + Y$ and $V = 2X - Y$. Calculate $\mathbb{E}(V|U)$. *(Hint: if W is independent of U, then $\mathbb{E}(W|U) = \mathbb{E}W$. If you choose $\lambda \in \mathbb{R}$ cleverly, then $W := V - \lambda U$ will be independent of U. (Since U and W are jointly Gaussian, to show independence it's enough to check that $Cov(U, W) = 0$.) Then write $V = \lambda U + W$.)*