Tools of Modern Probability

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Exercise sheet 3, fall 2019

3.1 *The Fatou lemma* is the following

Theorem 1 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and f_1, f_2, ... a sequence of measureabale functions $f_n : \Omega \to \mathbb{R}$, which are nonneagtive, e.g. $f_n(x) \geq 0$ for every $n = 1, 2, \dots$ and every $x \in \Omega$. Then*

$$\int_\Omega \liminf_{n\to\infty} f_n(x)\,\mathrm{d}\mu(x) \leq \liminf_{n\to\infty} \int_\Omega f_n(x)\,\mathrm{d}\mu(x)$$

(and both sides make sense).

Show that the inequality in the opposite direction is in general false, by choosing $\Omega = [0, 1]$, μ as the Lebesgue measure on $[0, 1]$, and constructing a sequence of nonnegative $f_n : [0, 1] \to \mathbb{R}$ for which $f_n(x) \xrightarrow{n\to\infty} 0$ for every $x \in [0, 1]$, but $\int_{[0,1]} f_n(x)\,\mathrm{d}x \geq 1$ for all n.

3.2 *Weak convergence and densities.* Prove the following

Theorem 2 *Let $\mu_1, \mu_2, \dots$ and μ be a sequence of probability distributions on $\mathbb{R}$ which are absolutely continouos w.r.t. Lebesgue measure. Denote their densities by $f_1, f_2, \dots$ and f, respectively. Denote their distribution functions by $F_1, F_2, \dots$ and F, respectively. Suppose that $f_n(x) \xrightarrow{n\to\infty} f(x)$ for every $x \in \mathbb{R}$. Then $F_n(x) \xrightarrow{n\to\infty} F(x)$ for every $x \in \mathbb{R}$.*

(Hint: Use the Fatou lemma to show that $F(x) \leq \liminf_{n\to\infty} F_n(x)$. For the other direction, consider $G(x) := 1 - F(x)$.)

3.3 Which of the spaces V below are linear spaces and why?

a.) $V := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_1 + 2x_2 = 0\}$, with the usual addition and the usual multiplication by a scalar.

b.) $V := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_1 + 2x_2 = 3\}$, with the usual addition and the usual multiplication by a scalar.

c.) $V := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_1 \geq 0\}$, with the usual addition and the usual multiplication by a scalar.

d.) $V := \{f : (0, 1) \to \mathbb{R} \,|\, f$ is continuous and $|f| \leq 100\}$, with the usual addition and the usual multiplication by a scalar.

e.) $V := \{f : (0, 1) \to \mathbb{R} \,|\, f$ is continuous and bounded$\}$, with the usual addition and the usual multiplication by a scalar.

3.4 On the linear spaces V and W below, which of the given transformations $T : V \to W$ are linear and why?

a.) $V = \mathbb{R}^3$, $W = \mathbb{R}^2$, $T((x_1, x_2, x_3)) := (x_1, x_2 + x_3)$.

b.) $V = \mathbb{R}^3$, $W = \mathbb{R}^2$, $T((x_1, x_2, x_3)) := (x_1, 1 + x_3)$.

c.) $V = \mathbb{R}^3$, $W = \mathbb{R}^2$, $T((x_1, x_2, x_3)) := (x_1, x_2x_3)$.

d.) $V := \{f : (-1, 1) \to \mathbb{R} \,|\, f$ differentiable$\}$, with the usual addition and the usual multiplication by a scalar; $W := \mathbb{R}$; $T(f) := f'(0)$.

3.5 On the linear spaces V below, which of the given two-variable functions $B : V \to \mathbb{R}$ are bilinear forms? Which ones are symmetric and positive definite? Why?

a.) $V = \mathbb{R}^3$, $B((x_1, x_2, x_3), (y_1, y_2, y_3)) := x_1y_2 + x_2y_3 + x_3y_1$

b.) $V = \mathbb{R}^2$, $B((x_1, x_2), (y_1, y_2)) := x_1x_2 + y_1y_2$

c.) $V = \mathbb{R}^2$, $B((x_1, x_2), (y_1, y_2)) := x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2$

d.) $V := \{f : [-1, 1] \to \mathbb{R} \,|\, f \text{ is differentiable}\}$, with the usual addition and the usual multiplication by a scalar; $B(f, g) := \int_{-1}^{1} x^2 f(x)g(x)\,\mathrm{d}x$

e.) $V := \{f : [-1, 1] \to \mathbb{R} \,|\, f \text{ is differentiable}\}$, with the usual addition and the usual multiplication by a scalar; $B(f, g) := \int_{-1}^{1} x f(x)g(x)\,\mathrm{d}x$

f.) $V := \{f : [-1, 1] \to \mathbb{R} \,|\, f \text{ is differentiable}\}$, with the usual addition and the usual multiplication by a scalar; $B(f, g) := \int_{-1}^{1} f'(x)g(x)\,\mathrm{d}x$

3.6 Let V be an inner product space. Show that the function $N : V \to \mathbb{R}$ defined as $N(x) := \sqrt{\langle x, x\rangle}$ is indeed a norm (usually denoted as $||x|| = N(x)$).

3.7 Let V be an inner product space, and let d denote the natural metric on it (defined as $d(x, y) := ||x - y||$). Let $x \in V$, let $D \subset V$ be convex, and assume that $d(x, D) = R > 0$ (where $d(x, D) := \inf\{d(x, y) \,|\, y \in D\}$ is the distance of x and D). Find a number $C \in \mathbb{R}$ (possibly depending on R) such that if $u, v \in D$, $d(x, u) \leq R + \varepsilon$ and $d(x, v) \leq R + \varepsilon$ with some $\varepsilon < R$, then $d(u, v) \leq C\sqrt{\varepsilon}$. *(Hint: estimate the length of the longest line segment that fits in the shell $\{y \in V \mid R \leq d(x, y) \leq R + \varepsilon\}$. A two-dimensional drawing will help.)*

3.8 Let V be an inner product space, and let d denote the natural metric (defined as $d(x, y) := ||x - y||$).

a.) Let $a, c, x \in V$ with $x \neq c$. Calculate the distance of a from the line $\{c + t(x - c) \,|\, t \in \mathbb{R}\}$ using $||a - c||$, $||x - c||$ and $\langle a - c, x - c\rangle$.

b.) Let $E \subset V$ be a linear subspace and let $a \in V$. Suppose that $c \in E$ is such that $d(a, x) \geq d(a, c)$ for every $x \in E$ – which means that c is the point in E which is closest to a. Prove that E is orthogonal to $a - c$, meaning that $\langle x, a - c\rangle = 0$ for every $x \in E$.

3.9 Let V be an inner product space over $\mathbb{R}$ and let $f : V \to \mathbb{R}$ be a linear form. Let $E := \{y \in V \,|\, f(y) = 0\}$ be the null-space of f. Suppose that $f(a) = 1$, $c \in E$ and $a - c$ is orthogonal to E, meaning $(a - c)y = 0$ for every $y \in E$. Now, for any $x \in V$, find the $\lambda \in \mathbb{R}$ for which $x_1 := x - \lambda(a - c) \in E$. Use this to get the relation between $f(x)$ and $(a - c)x$.

3.10 Represent the following functions $f : V \to \mathbb{R}$ as multiplication by a fixed vector, whenever this is possible due to the Riesz representation theorem.

a.) $V = \mathbb{R}^{10}$ with the usual inner product, $f((x_1, \dots, x_{10})) := x_5$ (evaluation at 5)

b.) $V = \mathbb{R}^{10}$ with the usual inner product, $f((x_1, \dots, x_{10})) := x_6 - x_5$ (discrete derivative at 5).

c.) $V = \mathbb{R}^{10}$ with the usual inner product, $f((x_1, \dots, x_{10})) := x_6 - 2x_5 + x_4$ (discrete second derivative at 5).

d.) $V = l^2 := \{x : \mathbb{N} \to \mathbb{R} \,|\, \sum_{i=1}^{\infty} x^2(i) < \infty\}$, with the inner product $x \cdot y := \sum_{i=1}^{\infty} x(i)y(i)$; $f(x) := \sum_{i=1}^{100} x(i)$.

e.) $V = l^2 := \{x : \mathbb{N} \to \mathbb{R} \,|\, \sum_{i=1}^{\infty} x^2(i) < \infty\}$, with the inner product $x \cdot y := \sum_{i=1}^{\infty} x(i)y(i)$; $f(x) := \sum_{i=1}^{\infty} x(i)$.

f.) $V = l^2 := \{x : \mathbb{N} \to \mathbb{R} \,|\, \sum_{i=1}^{\infty} x^2(i) < \infty\}$, with the inner product $x \cdot y := \sum_{i=1}^{\infty} x(i)y(i)$; $f(x) := \sum_{i=1}^{\infty} x^2(i)$.

g.) $V = L^2([0, 1]) := \{x : [0, 1] \to \mathbb{R} \,|\, \int_0^1 x^2(t)\,\mathrm{d}t < \infty\}$, with the inner product $x \cdot y := \int_0^1 x(t)y(t)\,\mathrm{d}t$; $f(x) := x(\frac{1}{2})$ (evaluation at $\frac{1}{2}$).

h.) $V = L^2([0,1]) := \{x : [0,1] \to \mathbb{R} \mid \int_0^1 x^2(t)\,\mathrm{d}t < \infty\}$, with the inner product $x \cdot y := \int_0^1 x(t)y(t)\,\mathrm{d}t$; $f(x) := x'(\frac{1}{2})$ (derivative at $\frac{1}{2}$).

i.) $V = L^2([0,1]) := \{x : [0,1] \to \mathbb{R} \mid \int_0^1 x^2(t)\,\mathrm{d}t < \infty\}$, with the inner product $x \cdot y := \int_0^1 x(t)y(t)\,\mathrm{d}t$; $f(x) := \int_{0.2}^{0.7} x(t)\,\mathrm{d}t$.

j.) $V = \{x : [0,1] \to \mathbb{R} \mid \int_0^1 x^2(t)\,\mathrm{d}t < \infty,\ f \text{ is differentiable}\}$, with the inner product $x \cdot y := \int_0^1 x(t)y(t)\,\mathrm{d}t$; $f(x) := x'(\frac{1}{2})$.

k.) $V = \{x : [0,1] \to \mathbb{R} \mid \int_0^1 x^2(t)\,\mathrm{d}t < \infty,\ f \text{ is continuous}\}$, with the inner product $x \cdot y := \int_0^1 x(t)y(t)\,\mathrm{d}t$; $f(x) := x(\frac{1}{2})$.

l.) $V = \{x : [0,1] \to \mathbb{R} \mid \int_0^1 x^2(t)\,\mathrm{d}t < \infty,\ f \text{ is continuous}\}$, with the inner product $x \cdot y := \int_0^1 x(t)y(t)\,\mathrm{d}t$; $f(x) := \int_{0.2}^{0.7} x(t)\,\mathrm{d}t$.