Banach spaces, Hilbert spaces

In analysis, we like spaces where a sequence, which <u>seems</u> to have a limit, <u>does</u> have a limit — in other words "taking the limit" does not lead 'out of the space': the limit of a convergent sequence is "also there".

This idea is formalized in the following definitions:

<u>Def</u> (Cauchy sequence):

Let (X, d) be a metric space, $x_1, x_2, x_3 \ldots \in X$.

The sequence $x_1, x_2, x_3 \ldots$ is called a <u>Cauchy sequence</u>, if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ such that

for any $n, m \geq N$ we have $d(x_n, x_m) < \varepsilon$.

[In words: If we throw away the first few (finitely many) elements, then all the remaining ones are closer than ε to each other.]

[Compare: in a convergent sequence, if we throw away the first few (finitely many) elements, then all the remaining ones are closer than ε to the limit.]

<u>Def</u> (complete metric space):

A metric space is called <u>complete</u> if every Cauchy sequence is convergent.

<u>Alternatively</u>: the <u>metric</u> is called complete.

<u>Examples</u>:

1.) $\mathbb{R}^n$, with the usual metric $d(x,y) := |x-y|$ is complete.

2.) $\mathbb{Q} := \{\text{rationals}\}$ with $d(x,y) := |x-y|$ is <u>not</u> complete:

e.g.

$$\left.\begin{array}{l} x_0 := 1 \\ x_1 := 1.4 \\ x_2 := 1.41 \\ x_3 := 1.414 \\ x_4 := 1.4142 \end{array}\right\} \begin{array}{l} x_n := \sqrt{2} \text{ rounded down} \\ \quad \text{to } n \text{ digits precision} \\ \text{so } x_n \in \mathbb{Q} \end{array}$$

is Cauchy, but has no limit in $\mathbb{Q}$.

3.) Any metric space can be "completed" in a standard way: "add" limit points of Cauchy sequences with algebraic trickery.

This is ~~the 1st~~ 1 of the standard constructions of $\mathbb{R}$ starting from $\mathbb{Q}$.

4)

<u>Def</u>: A normed space / inner product space is called <u>complete</u> if the induced metric is complete.

Def (Banach space): A complete normed space is called a Banach space.

Def (Hilbert space): A complete inner product space is called a Hilbert space.

Thm: For any measure space $(X, \mathcal{F}, \mu)$, the inner product space $L^2(\mu)$ is complete, so it's a Hilbert space.

[Actually: for any measure space $(X, \mathcal{F}, \mu)$ and any $1 \leq p \in \mathbb{R}$

Def: $L^p(\mu) := \{ f : X \to \mathbb{R} \mid \int_X |f|^p d\mu < \infty \}$, with functions equal μ-a.e. identified.

Thm: for $f \in L^p(\mu)$, $\|f\|_p := \sqrt[p]{\int_X |f|^p d\mu}$ is a norm,

so $L^p(\mu)$ with $\|.\|_p$ is a normed space.

[I will not prove this.]

Thm: Every L^p is complete, so a Banach space.

Remark: only L^2 is Hilbert, the other L^p norms are not induced by an inner product.]

Proof (plan) Let $f_1, f_2, \dots \in L^2(\mu)$ be a Cauchy sequence.

Show that the $L^2\text{-}\lim_{n\to\infty} f_n$ exists.

<u>Idea 1</u>: This is an abstract existence statement, but let's try to <u>construct</u> the limit!

<u>Idea 2</u>: L^2-convergence of functions is something mysterical. What we understand well is convergence of (sequences of) <u>numbers</u>.

So, couldn't it be $\boxed{\text{pointwise}}$ convergence?

<u>Problem</u>: L^2 convergence does <u>not</u> imply pointwise convergense, so even if we would know that $f_n \to 0$ in L^2, it would not mean $f_n \to 0$ a.e. ☹

Key idea: L^2 convergence does not imply convergence almost everywhere, but FAST L^2 convergence does.

<u>Practice</u>: Assume that $f_n \xrightarrow{L^2} 0$ <u>fast</u>, namely $\|f_n\|_2 < \frac{1}{2^n}$.

Show that $f_n \to 0$ μ-a.e. I mean $L^2(X,\mu)$. L^2 norm

<u>Solution</u>: ~~$f_n^2 \to 0$ To show~~ $f_n(x) \to 0$ is the same as $f_n^2(x) \to 0$. To show $f_n^2(x) \to 0$, it is (more than) enough to show that $G(x) := \sum_{n=1}^{\infty} f_n^2(x) < \infty$

To show $G(x) < \infty$ μ-a.e. x, it's ~~enough to~~ (more than) enough to show that $\int_X G(x) d\mu(x) < \infty$.

But this is obvious from the <u>monotone convergence</u> ~~thm~~ theorem: $G_N(x) := \sum_{n=1}^{N} f_n^2(x) \nearrow_{N \to \infty} G(x)$, so

$$\cancel{\text{and}} \quad \int_X G_N(x) d\mu(x) \xrightarrow{N \to \infty} \int_X G(x) d\mu(x),$$

$$\text{but } \int_X G_N(x)\, d\mu(x) = \sum_{n=1}^{N} \| f_n \|_2^2 \leq \sum_{n=1}^{N} \frac{1}{2^n} < 1 \qquad \square$$

[Remark: For $G(x) = \sum_{n=1}^{\infty} f_n^2(x)$, integrability also follows easily from ~~the~~ the Fubini theorem (or: Fubini-Tonelli): since $f_n^2(x) \geq 0$ for all n, x

$$\int_X G(x) d\mu(x) = \int_X \sum_{n=1}^{\infty} f_n^2(x) d\mu(x) \overset{\text{Fubini}}{=\!=} \sum_{n=1}^{\infty} \overbrace{\int_X f_n^2(x) d\mu(x)}^{\leq \frac{1}{2^n}} < 1$$
]

Proof of completeness of L^2

Let $f_1, f_2, f_3 \ldots \in L^2(X, \mu)$ be a Cauchy sequence.

<u>Step 1:</u> Choose a sub-sequence $f_{n_0}, f_{n_1}, f_{n_2}, f_{n_3} \ldots$ such that $\| f_{n_{k+1}} - f_{n_k} \|_2 \leq \frac{1}{2^{k+1}}$ $\forall k = 0, 1, \ldots$

(This can be done by induction) Trivial.

Step 2: set $g_k = f_{n_k} - f_{n_{k-1}}$ for $k = 1, 2, \dots$

so $f_{n_k} = f_{n_0} + \underbrace{g_1 + g_2 + \dots + g_k}_{G_k}$,

and $\|g_k\|_2 \leq \frac{1}{2^k}$.

To show convergence of f_{n_k}, it's enough to show that a.) $G(x) = \lim_{k\to\infty} G_k(x) = \sum_{\ell=1}^{\infty} g_\ell(x)$ exists μ-a.e. x, and

b.) G is also an L^2-limit of G_k.

~~Step 3:~~ $\boxed{\text{Both follow from the lemma below.}}$

Step 3: Now check that $f_n \xrightarrow{L^2} f := f_{n_0} + G$

not only along the subsequence, but along the entire sequence. (Trivial.)

Lemma: Let $g_1, g_2, g_3 \dots \in L^2(X, \mu)$ such that

$$\sum_{k=1}^{\infty} \|g_k\|_2 = A < \infty.$$

Then $G(x) := \sum_{k=1}^{\infty} g_k(x)$ exists for μ-a.e. $x \in X$

and $\|G\|_2 \leq A$

Proof of lemma:

let $H_n(x) = \sum_{k=1}^{n} |g_k(x)|$. This is increasing in n,

So $H(x) := \lim_{n\to\infty} H_n(x)$ ~~exs~~ exists, or $H_n(x) \nearrow H(x)$.

We claim that H is in L^2.

Indeed, $\|H_n\|_2 \le \sum_{k=1}^{n} \|g_k\|_2 \le A$,

$$\text{So } \int_X H_n^2(x)\,d\mu(x) \le A^2 \xRightarrow[\text{convergence theorem}]{\text{monotone}} \int_X H^2(x)\,d\mu(x) \le A^2.$$

So we even got that $\|H\|_2 \le A$.

Then of course $H(x) < \infty$ for μ-a.e x

$\Rightarrow$ the series $G(x) := \sum_{k=1}^{\infty} g_k(x)$ is $\boxed{\text{absolutely convergent}}$ for μ-a.e. x

$\Downarrow$

it is also convergent

and $|G(x)| \le H(x)$

$\Downarrow$

$\|G\|_2 \le \|H\|_2 \le A$ $\square$

Corollary: $G_n := \sum_{k=1}^{n} g_k \to G$ not only ~~in~~ μ-a.e, but also in L^2, since $G - G_n = \sum_{k=n+1}^{\infty} g_k \xRightarrow{\text{lemma}} \|G - G_n\|_2 \le \sum_{k=n+1}^{\infty} \|g_k\|_2 \xrightarrow{n\to\infty} 0$ $\square$

$\boxed{\text{COMPLETENESS OF } L^2 \text{ PROVEN.}}$

Special case:

let $X = \{1, 2, \dots, n\}$, $\mu = \mathcal{K}$ counting measure.

Then $\mathcal{L}^2(X, \mu) = \mathbb{R}^n$ with the usual Euclidean inner product $\langle \underline{x}, \underline{y} \rangle := x_1 y_1 + \dots + x_n y_n$

$\Rightarrow (\mathbb{R}^n, +, \cdot, \langle \, \rangle)$ is Hilbert.

Remark 1: The proof goes through without problem for any $p \geq 1$ to prove completeness of $\mathcal{L}^p$.

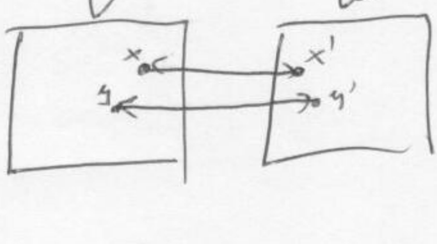
Remark about separable spaces

Def: let $(V, +, \cdot, \langle \, \rangle)$ and $(W, +, \cdot, \langle \, \rangle)$ be inner product spaces. These are called isomorphic, or: isometric, if $\exists$ a bijection $f: V \leftrightarrow W$ that preserves all the structure. Namely: with the notation $f(x) = x'$

$(x+y)' = x' + y' \quad \forall x, y \in V$

$(\lambda \cdot x)' = \lambda \cdot x' \quad \forall \lambda \in \mathbb{R}, x \in V$

$\langle x, y \rangle = \langle x', y' \rangle \quad \forall x, y \in V.$



Two isomorphic spaces are „essentially the same": one is the copy of the other.

<u>Def</u>: The metric space (X,d) is <u>separable</u> if it contains a countable dense set.

[That is: $\exists\, H \subset X$ such that H is countable and for any $x \in X$, $r > 0$ $\exists\, h \in H$ with $d(x,h) < r$.]

<u>Thm</u>: If $(V, +, \cdot, \langle\,\rangle)$ is a <u>separable</u> <u>Hilbert</u> space, then it is isomorphic

- either to $\mathbb{R}^n$ for some n
- or to ℓ^2.

In particular, all <u>infinite dimensional</u> <u>separable</u> <u>Hilbert</u> spaces are isomorphic to each other (including ℓ^2).

<u>Consequence</u>: In some Physics books, "the Hilbert space" refers to <u>the only</u> infinite dimensional separable Hilbert space (up to isometry).

<u>Remark</u>: Let $X = \mathbb{R}$, $\mu = \chi$ (counting measure).
Then $L^2(X,\mu)$ is <u>not</u> separable. (HW)