

Tools of Modern Probability
sample midterm exercise sheet, fall semester 2026-27
(working time: 90 minutes)

The midterm will contain 5 exercises. In this sample I give more: a possible midterm could consist of any 5 of these.

1. Describe the asymptotic behaviour of the following sequences as $n \rightarrow \infty$:

a.) $A_n := \int_0^{2\pi} \cos^{2n} x \, dx$

b.) $B_n := \int_{-1}^1 (x-1)^n (x+1)^n \, dx$

(By “describing the asymptotic behaviour” I mean: find sequences a_n and b_n given by nice simple formulas, such that $A_n \sim a_n$ and $B_n \sim b_n$.)

2. For $d = 1, 2, 3, \dots$ let c_d be the surface volume of the d -dimensional unit sphere

$$S_d := \{x \in \mathbb{R}^d \mid |x| = 1\}.$$

Show that $c_d = 2 \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$.

3. Describe the asymptotic behaviour of $\binom{2n}{n}$ as $n \rightarrow \infty$.
4. Let $\mathbb{P}$ be Lebesgue measure on $[0, 1] \times [0, 1]$, let $X : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ be given by $X(u, v) = \sqrt{u+v}$ and let $\mu = X_*\mathbb{P}$. Calculate $\int_{\mathbb{R}} x^2 \, d\mu(x)$.
5. Let μ be Lebesgue measure on $[-1, 1]$, let $f(x) = x^2$ and let ν have density f w.r.t. μ . Calculate $\int f \, d(f_*\nu)$.
6. Exercise 6.4 from the practice exercise sheet.
7. Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f, f_1, f_2, f_3, \dots : X \rightarrow \mathbb{R}$ be measurable functions such that $f_n(x) \rightarrow f(x)$ for μ -almost every $x \in X$ and $|f_n(x)| \leq 1$ for all x and n . In the following two special cases, show that

$$\int_X \lim_{n \rightarrow \infty} f_n(x) \, d\mu(x) \leq \lim_{n \rightarrow \infty} \int_X f_n \, d\mu,$$

or give a counterexample:

a.) $X = \mathbb{R}, \mu = Leb$

b.) $X = [0, 1]$ and μ is a probability measure on X .