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ESTIMATE OF THE HAUSDORFF MEASURE OF THE SIERPINSKI TRIANGLE

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Abstract. It is well-known that the Hausdorff dimension of the Sierpinski triangle Λ is $s = \log 3/\log 2$. However, it is a long standing open problem to compute the s-dimensional Hausdorff measure of Λ denoted by $\mathcal{H}^s(\Lambda)$. In the literature the best existing estimate is

$$0.670432 \leq \mathcal{H}^s(\Lambda) \leq 0.81794.$$

In this paper we improve significantly the lower bound. We also give an upper bound which is weaker than the one above but everybody can check it easily. Namely, we prove that

$$0.77 \leq \mathcal{H}^s(\Lambda) \leq 0.819161232881177$$

holds.

1. Introduction

In this paper we consider the Sierpinski triangle or gasket Λ. This is constructed as follows: take an equilateral triangle of side length equal to one, remove the inverted equilateral triangle of half length having the same center, then repeat this process for the remaining triangles infinitely many times as showed on Figs. 1, 2.

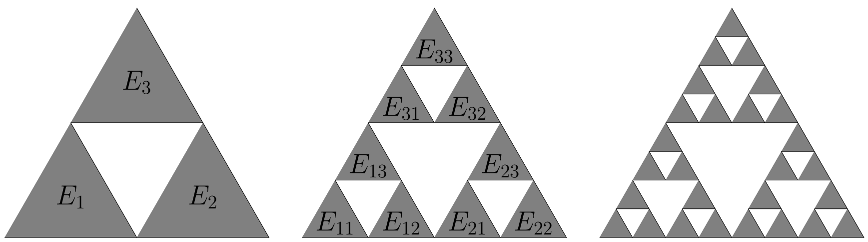

Figure 1. The triangles at the 1st, 2nd and the 3rd level

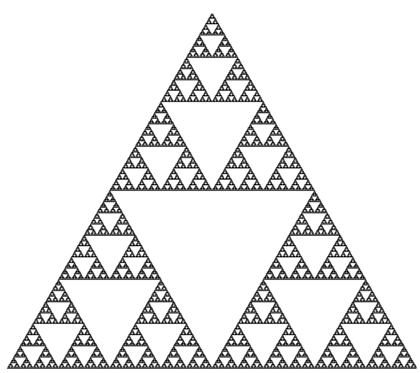

Figure 2. The Sierpinski triangle

In this paper we assumed that the diameter of the Sierpinski triangle is equal to 1. If the Sierpinski triangle is rescaled in such a way that its diameter is equal to t then the lower and upper bounds should be multiplied by $t^{\log 3/\log 2}$.

The Sierpinski triangle is one of the most famous fractals, and the Hausdorff dimension and measure are the most important characteristics of a fractal sets. The Sierpinski triangle is defined by an iterated function system, which satisfies a technical condition called the open set condition (OSC). Thus it follows from Hutchinson's Theorem [1] that the Hausdorff dimension is equal to $s = \log 3/\log 2$, and the s-dimensional Hausdorff measure $\mathcal{H}^s(\Lambda)$ of Λ is positive and finite. Since the Sierpinski triangle has an important role in many applications, it would be desirable to get a better understanding of its size. Therefore in the last two decades there have been a considerable attention paid to the computation of the s-dimensional Hausdorff measure of the Sierpinski triangle:

In 1987 Marion [2] showed that 0.9508 is an upper bound. In 1997 this was improved to 0.915, and later to 0.89 by Z. Zhou [3], [4]. In 2000 Z. Zhou and Li Feng proved that $\mathcal{H}^s(\Lambda) \leq 0.83078$ in [5]. The best upper bound is 0.81794, which was given by Wang Heyu and Wang Xinghua [6] in 1999 (in Chinese) with a computer algorithm.

In 2002 B. Jia, Z. Zhou and Z. Zhu [7] showed that 0.5 is a lower bound on the s-dimensional Hausdorff measure of the Sierpinski triangle. In 2004 R. Houjun and W. Weiyi [8] improved it to 0.5631. Finally, in 2006 B. Jia, Z. Zhou and Z. Zhu [9] proved that 0.670432 is a lower bound.

The main result of this paper is that $\mathcal{H}^s(\Lambda) \geq 0.77$.

The difficulty comes from geometry. The s-dimensional Hausdorff measure of Λ is defined by

$$\mathcal{H}^s(\Lambda) = \liminf_{\delta\to 0} \Big\{ \sum_k |A_k|^s, \text{ where } |A_k| < \delta \text{ and } A_k \text{ is a countable cover of } \Lambda \Big\}, \tag{1.1}$$

where $|A_k|$ denotes the diameter of the set A_k. When we estimate the Hausdorff measure we need to understand what is the most economical (in the sense of (1.1)) system of covers . Our most natural guess for this system is the covers by the level n triangles (the equilateral triangles on Fig. 1). However, this system of covers would result that the s-dimensional Hausdorff measure of Λ was equal to 1. On the other hand it is known that $\mathcal{H}^s(\Lambda) < 0.81794$. Therefore the best system of covers cannot possibly be the trivial one and this makes the problem difficult. To improve the existing best estimate on $\mathcal{H}^s(\Lambda)$ we use a Theorem of B. Jia. [10]. To state this Theorem we need to introduce some definitions.

It is well known (see [11]) that

$$\Lambda = \bigcup_{i=1}^{3} S_i(\Lambda),$$

where

$$S_1(x,y) = \left(\frac{1}{2}x, \frac{1}{2}y\right),$$

$$S_2(x,y) = \left(\frac{1}{2} + \frac{1}{2}x, 0 + \frac{1}{2}y\right),$$

$$S_3(x,y) = \left(\frac{1}{4} + \frac{1}{2}x, \frac{\sqrt{3}}{4} + \frac{1}{2}y\right).$$

Let E be the equilateral triangle of side length one with vertices: $(0,0)$, $(1,0)$, $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$. Now we define the level n triangles

$$E_{i_1\dots i_n} := S_{i_1\dots i_n}(E) = S_{i_1} \circ \dots \circ S_{i_n}(E)$$

for all $(i_1 \dots i_n) \in \{1,2,3\}^n$. Let μ be the uniform distribution measure on the Sierpinski triangle that is for all n and for all $i_1 \dots i_n$

$$\mu(E_{i_1\dots i_n}) = \frac{1}{3^n}.$$

After B. Jia we introduce the sequence

$$a_n = \min \frac{|\bigcup_{j=1}^{k_n} \Delta_j^{(n)}|^s}{k_n/3^n} = \min \frac{|\bigcup_{j=1}^{k_n} \Delta_j^{(n)}|^s}{\mu(\bigcup_{j=1}^{k_n} \Delta_j^{(n)})}, \tag{1.2}$$

where the minimum is taken for all non-empty sets of distinct level n triangles $\left\{\Delta_1^{(n)},\dots,\Delta_{k_n}^{(n)}\right\}$. It is easy to see that a_n is non-increasing (see [10]). Further B. Jia showed ([10]) that a_n is an upper bound on the Hausdorff measure of the Sierpinski triangle, and he also gave a lower bound using a_n :

Theorem 1 (B. Jia)**.** *The Hausdorff measure of the Sierpinski triangle satisfies:*

$$a_n e^{-\frac{16\sqrt{3}}{3}\cdot s\cdot\left(\frac{1}{2}\right)^n} \le \mathcal{H}^s(\Lambda) \le a_n \tag{1.3}$$

This Theorem implies that a_n tends to $\mathcal{H}^s(\Lambda)$.

Unfortunately there seems to be no way to compute a_n for $n \ge 6$. B. Jia [10] calculated a_1 and a_2. We can calculate a_3, a_4, a_5, but by (1.3) it results only that $\mathcal{H}^s(\Lambda) > 0.54$, which is not an improvement on the already existing lower bound. So instead of this direct approach we give a lower bound on a_n for every n. By using (1.3) this lower bound is also a lower bound on $\mathcal{H}^s(\Lambda)$. Using some complicated algorithm described in Sec. 4, 5 we point out that

$$a_n \geq 0.77$$

for all $n \in \mathbb{N}$. With Theorem 1 this implies that

$$\mathcal{H}^s(\Lambda) \geq 0.77.$$

We remind the reader that the best existing lower bond in the literature [9] was given in 2006: $\mathcal{H}^s(\Lambda) \geq 0.670432$.
Using the second inequality of Theorem 1, in Sec. 2 an upper bound is given on $\mathcal{H}^s(\Lambda)$ as follows: we provide a carefully selected collection of level 30 triangles $\left\{\Delta_1^{(30)}, \dots, \Delta_{k_{30}}^{(30)}\right\}$. This collection results an upper bound on a_{30} which in return gives the upper bound $\mathcal{H}^s(\Lambda) \leq 0.819161232881177$. In 1999 two Chinese mathematicians [6] published an upper bound which is better than this but their paper was published in Chinese giving in this way limited opportunity to check if their algorithm was correct.

Remark 1. *I want to thank my supervisor, Károly Simon for his support writing this article.*

2. Upper bound

In the definition of a_n (1.2) the minimum is taken for all non-empty sets of distinct level n triangles. We provide a collection of level n

triangles for all n, which gives an upper bound on a_n by definition, and an upper bound on $\mathcal{H}^s(\Lambda)$ by Theorem 1.

Take the following 6 points:

$$\{(1/4,0),(3/4,0),(1/8,\sqrt{3}/8),(3/8,3\sqrt{3}/8),(5/8,3\sqrt{3}/8),(7/8,\sqrt{3}/8)\}.$$

Let $D_1, D_2, \dots, D_6$ be the closed discs centered at these six points with radius 0.75. We write $D := D_1 \cap D_2 \cap \cdots \cap D_6$. Take all those level n triangles, which are contained in D (see Fig. 3 for an example). It is easy to see that the maximum distance between the chosen triangles will be exactly 0.75. Let us denote

$$c_n = \frac{0.75^{\log 3/\log 2}}{k_n/3^n},$$

where k_n is the number of the chosen level n triangles, which are in the region of intersection of the six discs.

The values for the c_n for small n are given by the following table:

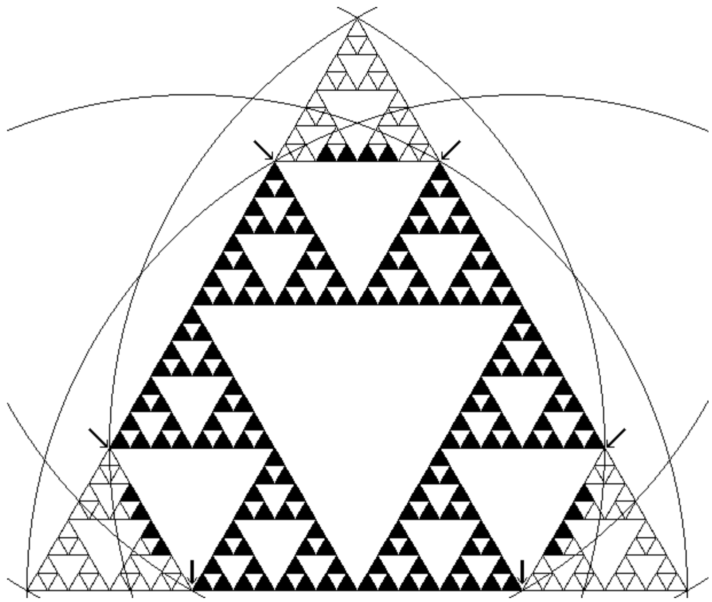

FIGURE 3. The black triangles are the chosen 174 level 5 triangles, so $c_5 = \frac{0.75^{\log 3/\log 2}}{174/3^5}$. Six arrows show the six given points.

n	Number of chosen triangles (k_n)	$k_n/3^n$	$\frac{0.75^{\log 3/\log 2}}{k_n/3^n}$
2	6	0.666666666666667	0.950753749115186
3	18	0.666666666666667	0.950753749115186
4	54	0.666666666666667	0.950753749115186
5	174	0.716049382716049	0.885184525038276
6	546	0.748971193415638	0.846275315146484
...			
28	17701192624554	0.773761997421774	0.819161234146210
29	53103577928148	0.773761998215679	0.819161233305724
30	159310733867010	0.773761998616697	0.819161232881177

Therefore using Theorem 1 we obtain that

$$c_{30} \geq a_{30} \geq \mathcal{H}^s(\Lambda)$$

holds. This implies:

Theorem 2. *The Hausdorff measure of the Sierpinski triangle is less than* 0.819161232881177.

One can show we cannot get a better upper bound on the s-dimension Hausdorff measure of the Sierpinski triangle than 0.819161232089868.

3. Lower bound, basic idea

For the convenience of the reader after giving the necessary definitions we are going to present a strongly simplified rough version of the idea of the algorithm. In Sec. 5. we will present the algorithm itself.

Definition 1. *Let $g > h$ be positive integers, and let $\Delta_1^{(g)}, \Delta_2^{(g)}, \dots \Delta_k^{(g)}$ be a set of distinct level g triangles, $\Delta_1^{(h)}, \Delta_2^{(h)}, \dots \Delta_l^{(h)}$ be a set of distinct level h triangles. We say that the set $\{\Delta_i^{(g)}\}_{i=1}^k$ is a* descendant *of the set $\{\Delta_j^{(h)}\}_{j=i}^l$ and we write $\{\Delta_j^{(h)}\}_{j=i}^l \overset{desc}{\longrightarrow} \{\Delta_i^{(g)}\}_{i=1}^k$, if* both *of the following conditions hold:*

- *For all $i \in \{1, 2, \dots k\}$ there is a j, such that $\Delta_i^{(g)} \subset \Delta_j^{(h)}$.*
- *For all $j \in \{1, 2, \dots l\}$ there is at least one i, such that $\Delta_i^{(g)} \subset \Delta_j^{(h)}$.*

See Fig. 4 for an example. This relation naturally defines a tree $\mathcal{T}$ for which the equilateral triangle E is the root. The set of level n nodes is equal to the set of all (non-empty) union of level n triangles. A level

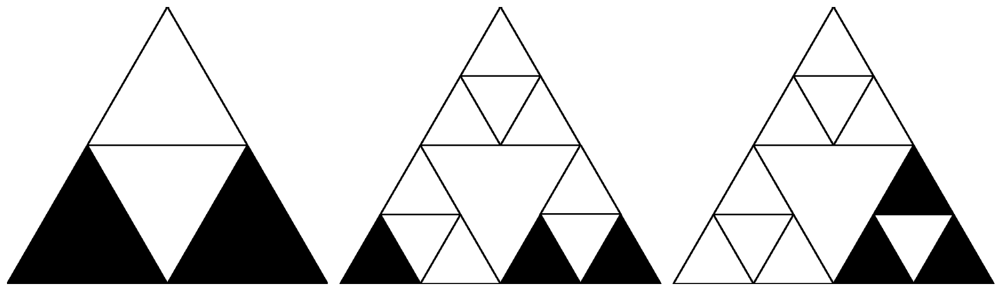

FIGURE 4. The set in the middle is a descendant of the left one, but the set on the right is *NOT* a descendant of the left one.

n node $\{\Delta_j^{(n)}\}_{j=1}^l$ is connected to a level $(n+1)$ node $\{\Delta_i^{(n+1)}\}_{i=1}^k$ if $\{\Delta_i^{(n+1)}\}_{i=1}^k$ is a descendant of $\{\Delta_j^{(n)}\}_{j=1}^l$. Figure 5 shows the top of the tree.

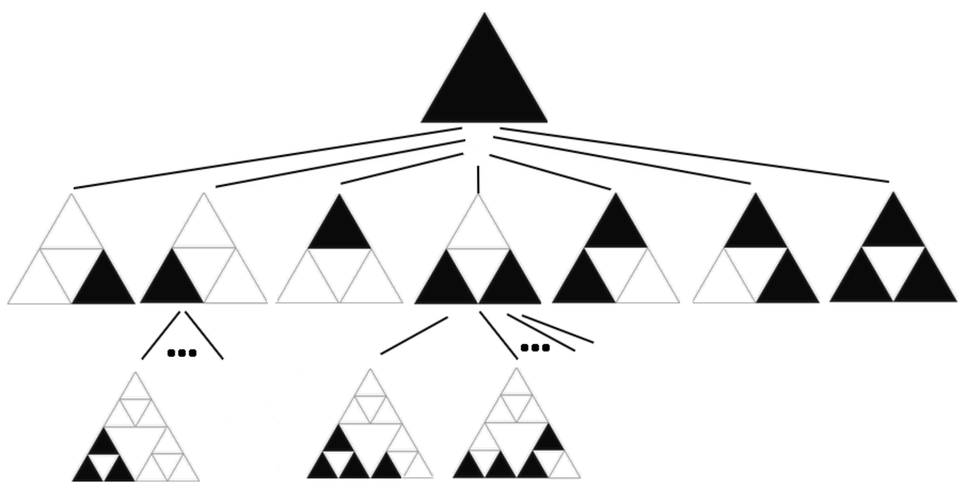

FIGURE 5. The top of the tree $\mathcal{T}$.

Let $v = \{\Delta_i^{(n)}\}_{i=1}^k$ be a level n node. Then we write $v^0 = n$ and we denote $\mathcal{T}_v$ the sub tree of $\mathcal{T}$ having v as root. ($\mathcal{T}_v$ consists of v and all those nodes w, which are descendant of v.) Let $E_v := \cup_{i=1}^k \Delta_i^{(n)}$.

We define

$$a_v := \frac{|E_v|^s}{k/3^n} = \frac{|E_v|}{\mu(E_v)}.$$

Our purpose is to give a lower bound on a_n (defined in (1.2)) for sufficiently large n, so we obtain a lower bound on its limit, $\mathcal{H}^s(\Lambda)$. It comes directly from the definitions that

$$a_n = \min_{v^\circ = n} a_v.$$

By using $a_n \downarrow \mathcal{H}^s(\Lambda)$ and taking infimum on both sides on n we obtain

$$\inf_{v \in \mathcal{T}} a_v = \lim_{n\to\infty} a_n = \mathcal{H}^s(\Lambda). \tag{3.1}$$

Let $v = \{\Delta_i^{(n)}\}_{i=1}^k$. We write

$$b_v := \max_{1\le i,j\le k} \min_{\underline{\mathbf{x}} \in \Delta_i^{(n)}, \underline{\mathbf{y}} \in \Delta_j^{(n)}} \frac{|\underline{\mathbf{x}} - \underline{\mathbf{y}}|^s}{k/3^n}.$$

Observe that for these $\underline{\mathbf{x}}$, $\underline{\mathbf{y}}$ we have

$$|\underline{\mathbf{x}} - \underline{\mathbf{y}}| = \max_{1\le i,j\le k} \operatorname{dist}(\Delta_i, \Delta_j).$$

Lemma 1. *The value b_v is a lower bound for a_w whenever $v \xrightarrow{desc} w$ holds. Namely,*

$$b_v \le \inf_{w\in\mathcal{T}_v} a_w.$$

Proof. For $v = \{\Delta_i^{(n)}\}_{i=1}^k$ let $w = \{\Delta_t^{(g)}\}_{t=1}^l \in \mathcal{T}_v$ be arbitrary. To give a lower bound on a_w first we give a lower bound on the diameter of E_w, then we give an upper bound on $\mu(E_w)$. We consider $\Delta_i^{(n)}$ and $\Delta_j^{(n)}$ for some $1 \le i, j \le k$. w is a descendant of v, so $E_w \cap \Delta_i^{(n)}$ and $E_w \cap \Delta_j^{(n)}$ are non-empty (see Fig. 6 for example). Thus the diameter of E_w is at

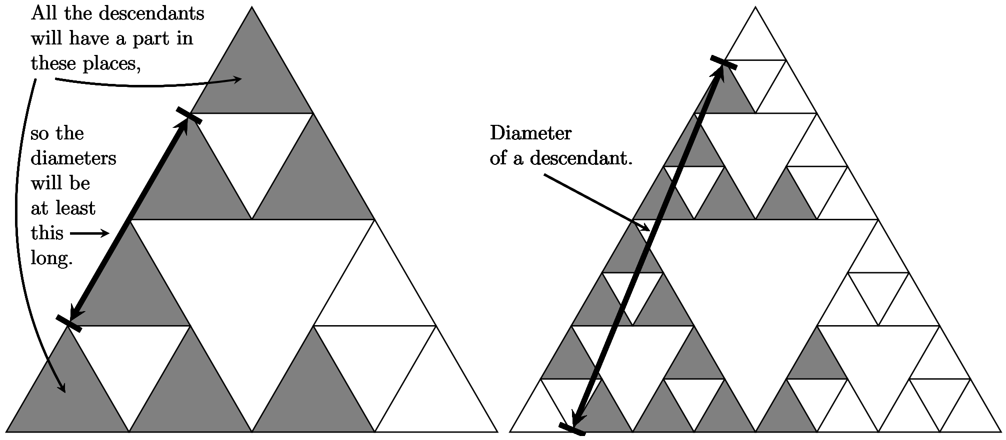

FIGURE 6. The set on the right is a descendant of the left one.

least

$$|E_w| > \min_{\underline{\mathbf{x}}\in\Delta_i^{(n)},\underline{\mathbf{y}}\in\Delta_j^{(n)}} |\underline{\mathbf{x}} - \underline{\mathbf{y}}| = \operatorname{dist}(\Delta_i^{(n)}, \Delta_j^{(n)}).$$

This inequality holds for all $1 \leq i, j \leq k$, so we can take the maximum over these pairs:

$$|E_w| > \max_{1\leq i,j\leq k} \min_{\underline{\mathbf{x}}\in\Delta_i^{(n)},\underline{\mathbf{y}}\in\Delta_j^{(n)}} |\underline{\mathbf{x}} - \underline{\mathbf{y}}|.$$

Because $v \xrightarrow{\text{desc}} w$, we have $E_w \subset E_v$. This yields

$$l/3^g = \mu(E_w) \leq \mu(E_v) = k/3^n,$$

therefore

$$b_v = \max_{1\leq i,j\leq k} \min_{\underline{\mathbf{x}}\in\Delta_i^{(n)},\underline{\mathbf{y}}\in\Delta_j^{(n)}} \frac{|\underline{\mathbf{x}} - \underline{\mathbf{y}}|^s}{k/3^n} \leq \frac{|E_w|^s}{l/3^g} = a_w.$$

□

First of all we give a lower bound only on a subtree defined by a finite set of nodes A. We define the set $\mathcal{T}_A$ as follows: $v \in \mathcal{T}_A$ if $v \in A$, or v is a descendant of a node w, which is in A. Namely,

$$\mathcal{T}_A = \bigcup_{w \in A} \mathcal{T}_w.$$

We write

$$B_A = \min_{v \in A} b_v.$$

We can apply the previous Lemma for every node in the set A, thus we have

$$B_A \le \inf_{v \in \mathcal{T}_A} a_v. \tag{3.2}$$

We are going to apply this inequality for the so called cross-sections. These are some subsets $C \subset \mathcal{T}$ of the nodes such that the lower bound on $a_v, v \in \mathcal{T}_C$ is a lower on the Hausdorff measure of the Sierpinski triangle. To make this definition precise first we define the set of the parents of C called P_C as

$$P_C = \{v \,|\, v \overset{\text{desc}}{\longrightarrow} w, w \in C\},$$

namely P_C is the set of nodes, which have a descendant in C.

Definition 2. *We call a finite set $C \subset \mathcal{T}$ a* cross-section*, if there exists a function φ, $\varphi : \mathcal{T} \setminus (\mathcal{T}_C \cup P_C) \to \mathcal{T}_C \cup P_C$ such that for every node $v \in \mathcal{T} \setminus (\mathcal{T}_C \cup P_C)$ we have*

$$a_{\varphi(v)} \le a_v,$$

and

$$\mu(E_{\varphi(v)}) \geq 3\mu(E_v).$$

Let v be a level n node. For a $k > n$ we write $\Gamma_k(v)$ for that level k descendant of v which has maximal μ measure. That is

$$\Gamma_k(v) = \{E_{i_1,\ldots i_k} \mid E_{i_1,\ldots i_k} \subset E_v\}.$$

See Fig. 7. We remark that

$$a_v = a_{\Gamma_k(v)}. \tag{3.3}$$

Namely, $|E_v| = |E_{\Gamma_k(v)}|$ and $\mu(E_v) = \mu(E_{\Gamma_k(v)})$ hold.

Fact 1. *Let H be an arbitrary subset of $\mathcal{T}$.*

Then for every $k \geq 0$ we have

$$\inf_{v\in\mathcal{T}_H} a_v = \inf_{v\in\mathcal{T}_H \bigcap\{w \,|\, w^\circ \geq k\}} a_v. \tag{3.4}$$

Proof. It is enough to verify that

$$\inf_{v\in\mathcal{T}_H} a_v \geq \inf_{v\in\mathcal{T}_H \bigcap\{w \,|\, w^\circ \geq k\}} a_v$$

holds. To do so, let $u \in \mathcal{T}_H \setminus \{w \,|\, w^\circ \geq k\}$ be arbitrary. By using $\Gamma_k(u) \in \mathcal{T}_H \bigcap \{w \,|\, w^\circ \geq k\}$ and (3.3) we have

$$a_u = a_{\Gamma_k(u)} \geq \inf_{v\in\mathcal{T}_H \bigcap\{w \,|\, w^\circ \geq k\}} a_v,$$

which completes the proof. □

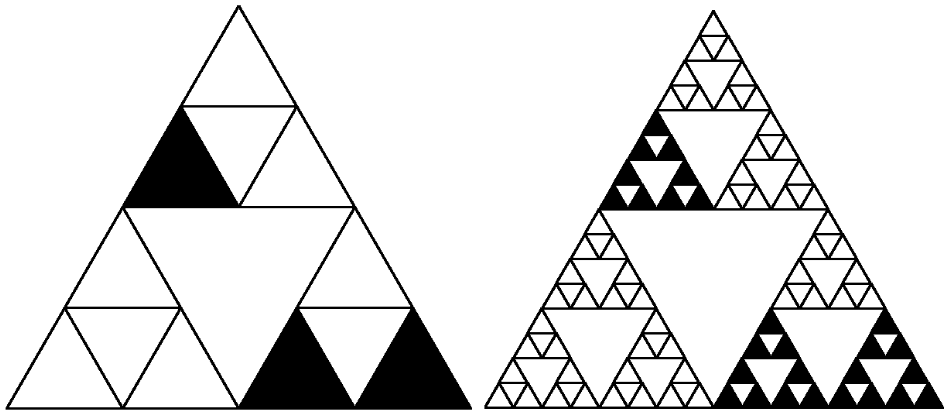

FIGURE 7. The node v on the left is a level 2 node, the node on the right is the node $\Gamma_4(v)$.

For a cross-section C we define

$$B_C = \min_{v \in C} b_v.$$

Lemma 2. *For every cross-section C we have*

$$\inf_{v \in \mathcal{T}} a_v = \inf_{v \in \mathcal{T}_C} a_v, \tag{3.5}$$

and

$$B_C \leq \inf_n a_n = \mathcal{H}^s(\Lambda). \tag{3.6}$$

Proof. It is easy to see that (3.6) is an immediately consequence of (3.5). Namely, using (3.1) and (3.2) we have

$$B_C \leq \inf_{v \in \mathcal{T}_C} a_v = \inf_{v \in \mathcal{T}} a_v = \inf_n a_n = \mathcal{H}^s(\Lambda)$$

Let M_C be the maximum level of the nodes which are contained in the set C. We define

$$K_{M_C} = \left\{ v \,|\, v^0 \geq M_C \right\}.$$

Using Fact 1 we have

$$\inf_{v\in\mathcal{T}} a_v = \inf_{v\in\mathcal{T}\cap K_{M_C}} a_v, \quad \text{and} \quad \inf_{v\in\mathcal{T}_C} a_v = \inf_{v\in\mathcal{T}_C\cap K_{M_C}} a_v.$$

Thus to prove (3.5) it is enough to verify that

$$\inf_{v\in\mathcal{T}\cap K_{M_C}} a_v = \inf_{v\in\mathcal{T}_C\cap K_{M_C}} a_v \tag{3.7}$$

holds.

We fix a $v \in K_{M_C} \setminus \mathcal{T}_C$. To verify (3.7) we will show that there exists a node $t \in K_{M_C} \cap \mathcal{T}_C$ such that

$$a_v \geq a_t \tag{3.8}$$

holds. Since $K_{M_C} \cap P_C = \emptyset$, thus $v \in \mathcal{T} \setminus (\mathcal{T}_C \cup P_C)$. C is a cross-section, by definition there exists φ such that $\varphi(v) \in \mathcal{T}_C \cup P_C$. If $\varphi(v) \in K_{M_C}$, then $\varphi(v) \in \mathcal{T}_C$ as well, so (3.8) follows from choosing $t = \varphi(v)$ and by using

$$a_v \geq a_{\varphi(v)}.$$

If $\varphi(v) \in P_C$, then let us consider $\Gamma_{M_C}(\varphi(v))$. If $\Gamma_{M_C}(\varphi(v)) \in \mathcal{T}_C$ then $t := \Gamma_{M_C}(\varphi(v))$ yields (3.8). If $\Gamma_{M_C}(\varphi(v)) \notin \mathcal{T}_C$ then by (3.3) and by the definition of φ and Γ_{M_C} we have

$$\begin{aligned} a_v \geq a_{\varphi(v)} &= a_{\Gamma_{M_C}(\varphi(v))} \\ \mu(E_{\Gamma_{M_C}(\varphi(v))}) &\geq 3\mu(E_v) \\ \Gamma_{M_C}(\varphi(v)) &\in K_{M_C} \setminus \mathcal{T}_C \end{aligned} \tag{3.9}$$

So, we can repeat the same for the node $w_1 := \Gamma_{M_C}(\varphi(v))$ instead of v. If $\Gamma_{M_C}(\varphi(w_1)) \in \mathcal{T}_C$ then we are ready as we saw above. If not then (3.9) holds for w_1 instead of v. Note that this follows that $0 < 9\mu(E_v) \leq \mu(E_{\Gamma_{M_C}(\varphi(w_1))}) \leq 1$. This shows that there must exists a finite N such that $\Gamma_{M_C}(\varphi(w_N)) \in \mathcal{T}_C$, where $w_{k+1} := \Gamma_{M_C}(\varphi(w_k))$. This completes the proof of (3.8)

□

Take the following set:

$$C_0 = \{v \,|\, v^0 = 2, v \in \cup\mathcal{T}_{\{E_1,E_2\}} \cup \mathcal{T}_{\{E_1,E_3\}} \cup \mathcal{T}_{\{E_2,E_3\}}, \tag{3.10}$$

$$v \neq \{E_{1,2}, E_{2,1}\}, v \neq \{E_{1,3}, E_{3,1}\}, v \neq \{E_{2,3}, E_{3,2}\}\} \cup \{\{E_1, E_2, E_3\}\}.$$

See Fig. 1 for labelling. There are $7 \cdot 7 = 49$ descendants of the node $\{E_1, E_2\}$ at level 2. Counting the same for $\{E_1, E_3\}$ and $\{E_2, E_3\}$ we have $3 \cdot 49 = 147$ nodes. Let us remove the nodes $\{E_{1,2}, E_{2,1}\}$, $\{E_{1,3}, E_{3,1}\}$, $\{E_{2,3}, E_{2,3}\}$, and take the node $\{E_1, E_2, E_3\}$, so we get the set C_0. Thus C_0 consists of $147 - 3 + 1 = 145$ nodes.

Proposition 1. *The set C_0 is a cross-section.*

Proof. Note that

$$\mathcal{T} \setminus (\mathcal{T}_{C_0} \cup P_{C_0}) = \mathcal{T}_{E_1} \cup \mathcal{T}_{E_2} \cup \mathcal{T}_{E_3} \cup \mathcal{T}_{\{E_{1,2},E_{2,1}\}} \cup \mathcal{T}_{\{E_{1,3},E_{3,1}\}} \cup \mathcal{T}_{\{E_{2,3},E_{2,3}\}}.$$

To define the function φ in the Definition 2, first we define an auxiliary function $\psi : \mathcal{T} \setminus (\mathcal{T}_{C_0} \cup P_{C_0}) \to \mathcal{T}$. (See Figs. 8 and 9.)

- For $v \in \mathcal{T}_{\{E_j\}}$, where $j = 1, 2, 3$ let

$$\psi(v) := \{E_{i_2,i_3,\dots,i_n} \mid E_{j,i_2,i_3,\dots,i_n} \in v\},$$

- for $v \in \mathcal{T}_{\{E_{1,2},E_{2,1}\}}$ let

$$\psi(v) := \{E_{1,i_1,i_2,\dots,i_n} \mid E_{1,2,i_1,i_2,\dots,i_n} \in v\} \cup \{E_{2,i_1,i_2,\dots,i_n} \mid E_{2,1,i_1,i_2,\dots,i_n} \in v\},$$

- for $v \in \mathcal{T}_{\{E_{1,3},E_{3,1}\}}$ let

$$\psi(v) := \{E_{1,i_1,i_2,\dots,i_n} \mid E_{1,3,i_1,i_2,\dots,i_n} \in v\} \cup \{E_{3,i_1,i_2,\dots,i_n} \mid E_{3,1,i_1,i_2,\dots,i_n} \in v\},$$

- for $v \in \mathcal{T}_{\{E_{2,3},E_{3,2}\}}$ let

$$\psi(v) := \{E_{2,i_1,i_2,\dots,i_n} \mid E_{2,3,i_1,i_2,\dots,i_n} \in v\} \cup \{E_{3,i_1,i_2,\dots,i_n} \mid E_{3,1,i_1,i_2,\dots,i_n} \in v\}.$$

Clearly,

$$|E_{\psi(v)}| = 2|E_v|$$

and

$$\mu(E_{\psi(v)}) = 3\mu(E_v).$$

Thus we have

$$a_{\psi(v)} = a_v.$$

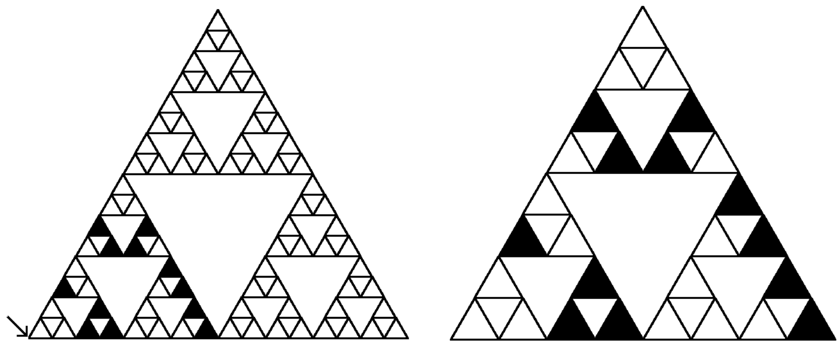

FIGURE 8. The node v on the left is a descendant of the node $\{E_1\}$, the node on the right is $\psi(v)$. The arrow shows the fix point of rescaling.

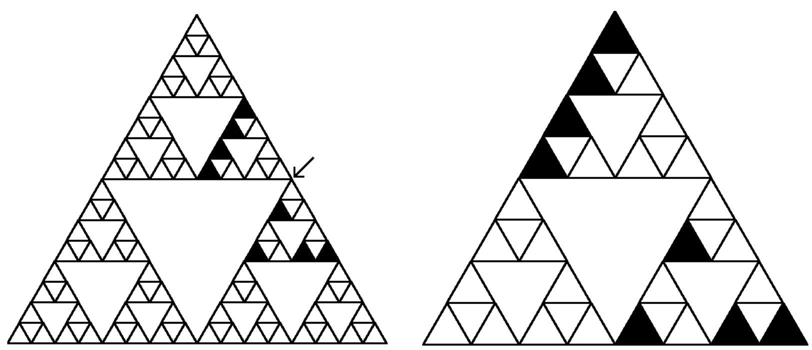

FIGURE 9. The node v on the left is a descendant of the node $\{E_{2,3}, E_{3,2}\}$, the node on the right is $\psi(v)$. The arrow shows the fix point of rescaling.

This follows that for every $v \in \mathcal{T} \setminus (\mathcal{T}_{C_0} \cup P_{C_0})$ there exists an N such that C_0 is a cross-section with the function

$$\varphi(v) := \psi^N(v) = \underbrace{\psi \circ \cdots \circ \psi}_{N}(v) \in \mathcal{T}_{C_0} \cup P_{C_0}.$$

(See Fig. 10.)

$\square$

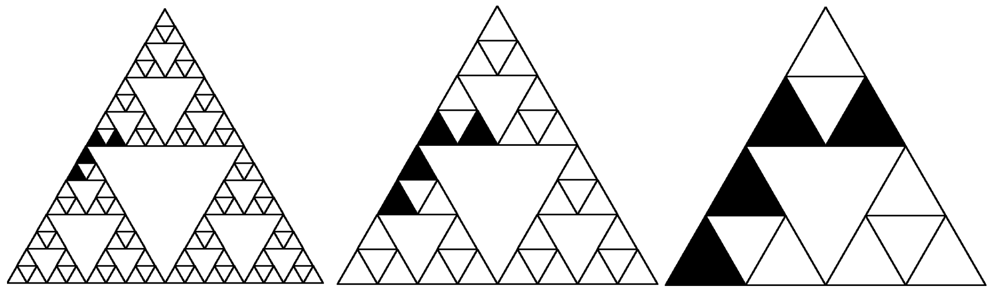

FIGURE 10. The node v on the left is a descendant of the node $\{E_{1,3}, E_{3,1}\}$. The node in the middle is $\Psi(v)$. The node on the right is $\varphi(v) = \Psi(\Psi(v))$.

For the convenience of the reader we present a simplified algorithm for choosing cross-sections C_n in the next Sec. Finally, in Sec. 5 we improve this algorithm significantly by using symmetries and a convexity argument.

4. Algorithm

Our purpose is to choose cross-sections C_n in such a way that B_{C_n} gets as large as possible, but a computer can check it in acceptable length of time. It is a natural idea to choose a starting cross-section, and modify it in hope to get a better lower bound. For $n = 0$ take the set C_0 defined in (3.10). For every n in the n-th step find a node $v \in C_n$ where

$$b_v = \min_{w \in C_n} b_w.$$

To obtain C_{n+1} from C_n we throw away v from C_n and we add to C_n all the next level descendants of v. It follows from the definition of b_v that $B_{C_{n+1}} \geq B_{C_n}$.

The following algorithm consists of three steps. It gives a lower bound on the s-dimensional Hausdorff measure of the Sierpinski triangle every time it reaches Step 2. It will run forever, but during its running it will give better and better lower bounds.

Algorithm 1. Step 1. *Start with the set* C_0 *from the previous Section. Let* $n := 0$.

Step 2. *Find* $\min_{v \in C_n} b_v$. *Below we prove that* C_n *is a cross-section. So, it follows from Lemma 2 that we have*

$$\min_{v \in C_n} b_v \leq \mathcal{H}^s(\Lambda).$$

Step 3. *Find a node* $v \in C_n$ *for which* $b_v = \min_{w \in C_n} b_w$ *(if such a* v *is not unique, then choose any of them). Let us suppose* v *is level* m *node. We define* S_n *as the set of all of those level* $m+1$ *nodes, which are descendants of the node* v. *That is*

$$S_n := \{w \mid w^0 = m+1,\, v \overset{desc}{\longrightarrow} w\},$$

Let

$$C_{n+1} := S_n \cup C_n \setminus \{v\}.$$

Increase n *by* 1. *Go to Step 2.*

Above we used the fact that C_n is a cross-section for every n. This is so because we have already seen that C_0 is a cross-section and

$$\mathcal{T}_{C_{n+1}} \cup P_{C_{n+1}} = \mathcal{T}_{C_n} \cup P_{C_n}$$

holds for all n.

5. Lower bound, making the algorithm faster

Our aim here is to improve the algorithm presented in the previous Section. To do so, for every n we define a cross section Q_n. Namely, let $Q_0 := C_0$. Assume that Q_n is already defined. To define Q_{n+1} first we define a certain set of nodes $D_n \subset Q_n$ as it is detailed later in the Section. It is important that the set D_n is much smaller than Q_n. We choose a $v \in D_n$ for which

$$b_v = \min_{w \in D_n} b_w. \tag{5.1}$$

Then the special choice of D_n will guarantee that

$$b_v \leq \mathcal{H}^s(\Lambda). \tag{5.2}$$

To get Q_{n+1} we replace v (defined in (5.1)) with its next level descendants.

To define D_n we need to introduce the notion of the convexity of a node. We remark that D_n will consist only of convex nodes.

Definition 3. *Let v be a level n node. We write*

$$\mathit{conv}(v) = \{E_{i_1,i_2,\ldots,i_n} \,|\, E_{i_1,i_2,\ldots,i_n} \textit{ is contained in the convex hull of } E_v\}.$$

See Fig. 11. for an example. We call a node v *convex*, if $v = \mathit{conv}(v)$, otherwise we call it non-convex.

Lemma 3. *Let v be a non-convex level n node. If v' is a level m descendant of the node v, and $\Theta \in \mathit{conv}(v) \setminus v$ is a level n triangle, then the closed convex hull of $E_{v'}$ intersects Θ.*

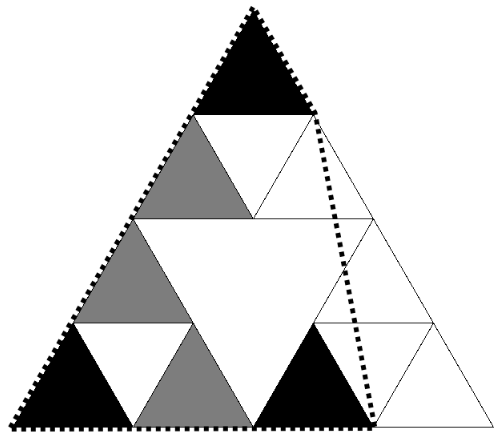

FIGURE 11. The node v consists of the black triangles. The convex hull of E_v is showed with dashed lines. The black and the gray triangles together form the node $conv(v)$.

Proof. We assume that $v = \{\Delta_i\}_{i=1}^{k}$. By definition of convexity we have

$$\Theta \subset \left\{ \sum_{i=1}^{k} \alpha_i \underline{\mathbf{x_i}} \,|\, \underline{\mathbf{x_i}} \in \Delta_i, \alpha_i \geq 0, \sum_{i=1}^{k} \alpha_i = 1 \right\}.$$

For $i = 1, 2, \ldots, k$ let $\underline{\mathbf{t_i}} \in \Delta_i$ be arbitrary points. To verify the assertion of the Lemma it is enough to show that

$$\Theta \cap \left\{ \sum_{i=1}^{k} \alpha_i \underline{\mathbf{t_i}} \,|\, \alpha_i \geq 0, \sum_{i=1}^{k} \alpha_i = 1 \right\} \neq \emptyset \tag{5.3}$$

holds for every choice of $\underline{\mathbf{t_1}}, \ldots, \underline{\mathbf{t_k}}$. We prove it by contradiction. Let us suppose there exist $\underline{\mathbf{t_1}}, \ldots, \underline{\mathbf{t_k}}$ such that (5.3) does not hold. Then there exists a line e, such that e separates Θ and the convex hull of $\underline{\mathbf{t_1}}, \ldots, \underline{\mathbf{t_k}}$. Let $\underline{\mathbf{a}}$ be one of the normal unit vectors of e. Put $r := \underline{\mathbf{z}} \cdot \underline{\mathbf{a}}$, where $\underline{\mathbf{z}} \in e$ arbitrary, and dot means the scalar product. Let us define

$$q := \max_{\underline{\mathbf{x}}, \underline{\mathbf{y}} \in \Theta} (\underline{\mathbf{x}} - \underline{\mathbf{y}}) \cdot \underline{\mathbf{a}}. \tag{5.4}$$

Without loss of generality we may assume that

$$\max\left\{\left(\sum_{i=1}^{k}\alpha_i\underline{\mathbf{t_i}}\right)\cdot\underline{\mathbf{a}}\,\middle|\,\alpha_i\geq 0,\sum_{i=1}^{k}\alpha_i=1\right\}<r$$

and

$$\min_{\underline{\mathbf{x}}\in\Theta}\underline{\mathbf{x}}\cdot\underline{\mathbf{a}}>r$$

hold, otherwise take $-\underline{\mathbf{a}}$ instead of $\underline{\mathbf{a}}$. The last inequality and (5.4) implies that

$$\max_{\underline{\mathbf{x}}\in\Theta}\underline{\mathbf{x}}\cdot\underline{\mathbf{a}}>q+r,$$

let us denote $\underline{\mathbf{x_0}}$ where the maximum is attained. Since $\Theta\in conv(v)\setminus v$, thus for $i=1,2,\dots,k$ there exist $\underline{\mathbf{u_i}}\in\Delta_i$, and $\beta\geq 0$ such that

$$\underline{\mathbf{x_0}}=\sum_{i=1}^{k}\beta_i\underline{\mathbf{u_i}}$$

holds. Using the fact all level n triangles are translations of each others and using (5.4), for $i=1,2,\dots,k$ we have

$$(\underline{\mathbf{u_i}}-\underline{\mathbf{t_i}})\cdot\underline{\mathbf{a}}\leq q.$$

Observe that

$$q+r<\underline{\mathbf{x_0}}\cdot\underline{\mathbf{a}}=\left(\sum_{i=1}^{k}\beta_i\underline{\mathbf{t_i}}\right)\cdot\underline{\mathbf{a}}+\left(\sum_{i=1}^{k}\beta_i(\underline{\mathbf{u_i}}-\underline{\mathbf{t_i}})\right)\cdot\underline{\mathbf{a}}<r+q,$$

which is a contradiction, and completes the proof.

□

The next Lemma shows that for any descendant w of a non-convex node v the value of a_w can be at most slightly bigger than some of the same level descendants of $conv(v)$. We will need this to verify (5.2).

Lemma 4. *Let v be a non-convex level n node and let $m > n$ be arbitrary. If $v \xrightarrow{desc} v'$, $v'^{\circ} = m$, then there exists a node w', $w'^{\circ} = m$, $conv(v) \xrightarrow{desc} w'$, such that*

$$|E_{w'}| \leq |E_{v'}| + \frac{2}{2^m} \tag{5.5}$$

and

$$E_{v'} \subset E_{w'}$$

hold.

Proof. We write $v = \{\Delta_i\}_{i=1}^k$, and $conv(v) \setminus v = \{\Theta_j\}_{j=1}^l$. Let us define the polygon H as the closed convex hull of $E_{v'}$. We proved in Lemma 3 that Θ_j intersects H for $1 \leq j \leq l$. If the polygon H intersects a triangle Θ_j, then for all j there exists at least one level m triangle $\Theta'_j \subset \Theta_j$, such that Θ'_j intersects H as well. We write $\underline{\mathbf{t_j}}$ for a point where H intersects the triangle Θ'_j. Let

$$w' = v' \cup \{\Theta'_j\}_{j=1}^l.$$

Let $\underline{\mathbf{q_0}}, \underline{\mathbf{w_0}} \in E_{w'}$ be some points where the maximum

$$|E_{w'}| = \max_{\underline{\mathbf{q}}, \underline{\mathbf{w}} \in E_{w'}} |\underline{\mathbf{q}} - \underline{\mathbf{w}}|,$$

is attained. If $\underline{\mathbf{q_0}}, \underline{\mathbf{w_0}} \in E_{v'}$ then we have $|E_{w'}| = |\underline{\mathbf{q_0}} - \underline{\mathbf{w_0}}| \leq |E_{v'}|$, thus the inequality (5.5) holds. If one of them is not in $E_{v'}$, let say $\underline{\mathbf{q_0}}$, then there exists a j such that $\underline{\mathbf{q_0}} \in \Theta'_j$ and $\underline{\mathbf{w_0}} \in E_{v'}$. Using triangle inequality we have

$$|E_{w'}| = |\underline{\mathbf{q_0}} - \underline{\mathbf{w_0}}| \leq |\underline{\mathbf{q_0}} - \underline{\mathbf{t_j}}| + |\underline{\mathbf{t_j}} - \underline{\mathbf{w_0}}| \leq |\Theta'_j| + |E_{v'}| = \frac{1}{2^m} + |E_{v'}|$$

because $\underline{\mathbf{q_0}}, \underline{\mathbf{t_j}} \in \Theta'_j$. If both $\underline{\mathbf{q_0}}$ and $\underline{\mathbf{w_0}}$ are not in $E_{v'}$, then using triangle inequality twice we have $|E_{w'}| \leq \frac{2}{2^m} + |E_{v'}|$. □

The following Lemma helps us to reduce the number of cases to be checked in an analogous way to the previous Lemma.

Lemma 5. *Let $v = \{\Delta_i\}_{i=1}^k$ be a level n node, $\Delta = E_{i_1,\dots,i_n}$ be a level n triangle such that $\Delta \notin v$. Further, let x be one of the vertices of the triangle Δ. We write $D(x, r)$ for the closed disc centered at x with radius r. If*

$$E_v \subseteq D(x, \max_{1 \leq i,j \leq k} dist(\Delta_i, \Delta_j))$$

holds then for all level m descendant v' of the node v there exists a level m triangle $\Delta' \subset \Delta$, such that

$$|E_{v'} \cup \Delta'| \leq |E_{v'}| + \frac{1}{2^m}$$

Proof. Let Δ' be that level m triangle, which has x as one of its vertices, and $\Delta' \subset \Delta$. As we saw in the proof of Lemma 1, $\max_{1 \leq i,j \leq k} \mathrm{dist}(\Delta_i, \Delta_j))$ is a lower bound on $|E_{v'}|$. Furthermore,

$$|E_{v'} \cup \Delta'| = \max_{\underline{\mathbf{q}}, \underline{\mathbf{w}} \in E_{v'} \cup \Delta'} |\underline{\mathbf{q}} - \underline{\mathbf{w}}|.$$

Let $\underline{\mathbf{q_0}}, \underline{\mathbf{w_0}}$ be those points where this maximum is attained. Either $\underline{\mathbf{q_0}}, \underline{\mathbf{w_0}} \in E_{v'}$, or one of the points, let us say $\underline{\mathbf{q_0}} \in \Delta'$, and $\underline{\mathbf{w_0}} \in E_{v'}$.

By using $|\Delta'| = 1/2^m$ and triangle inequality both cases implies the statement.

□

The following Theorem will show (with $D = D_n$) how the sequence of sets $\{D_n\}_{n=0}^{\infty}$ mentioned in the introduction of this Section gives us a lower bound on the Hausdorff measure $\mathcal{H}^s(\Lambda)$. Then after this theorem we will construct $\{D_n\}_{n=0}^{\infty}$.

Theorem 3. *Let $Q \subset \mathcal{T}$ be a cross-section. We choose an arbitrary $D \subset Q$ which satisfies the following assumption:*
For all $v \in Q \setminus D$ there exists a node $w \in \mathcal{T}_Q \cup P_Q$, such that

- $w^\circ = v^\circ$,
- $E_v \subsetneq E_w$,
- *for* $v \xrightarrow{desc} v'$ *there exists a* $w \xrightarrow{desc} w'$ *with* $v'^\circ = w'^\circ =: m$ *such that*

$$E_{v'} \subset E_{w'} \quad \text{and} \quad |E_{w'}| \leq |E_{v'}| + \frac{2}{2^m}.$$

Then

$$B_D = \min_{t \in D} b_t \leq \mathcal{H}^s(\Lambda).$$

Proof. Let us denote the finitely many elements of $Q \setminus D$ by:

$$Q \setminus D = \{v_1, v_2, \ldots, v_k\}.$$

Let $\varepsilon > 0$ be arbitrary. Choose an $M > \max_{v \in Q} v^\circ$ which also satisfies

$$\left(1 + \frac{2}{\delta 2^M}\right)^{k \cdot s} < 1 + \varepsilon,$$

where

$$\delta := \inf_{v \in \mathcal{T}_{C_0}} |E_v|.$$

We remind the reader that C_0 was defined in (3.10). It is easy to see that $\delta > 0$. Recall that $K_M = \{v \,|\, v^0 \geq M\}$. Fix an arbitrary $v'_0 \in K_M \cap \mathcal{T}_Q$. To prove the assertion of the Theorem, it is enough to show that

$$a_{v'_0} \geq \frac{B_D}{1+\varepsilon}. \tag{5.6}$$

Namely,

$$\mathcal{H}^s(\Lambda) = \inf_{v \in \mathcal{T}} a_v = \inf_{v \in \mathcal{T}_Q} a_v = \inf_{v \in K_M \cap \mathcal{T}_Q} a_v \geq \frac{B_D}{1+\varepsilon},$$

here we used first (3.1) then (3.5) and at the third equality we used Fact 1.

Now we define by mathematical induction a finite (at least one and at most k elements) sequence of nodes

$$v'_0, v'_1, \ldots, v'_l,$$

where $v'_l \in \mathcal{T}_D$ and $v'_0, v'_1, \ldots, v'_{l-1} \in \mathcal{T}_Q \setminus \mathcal{T}_D$. Namely, assume that we have already defined v'_n for an $n \geq 0$. If $v'_n \in \mathcal{T}_D$, then let $v'_l = v'_n$ be the last element of the sequence. Otherwise $v'_n \in \mathcal{T}_Q \setminus \mathcal{T}_D$, so there exists a node $v_{i_n} \in \{v_1, v_2, \ldots, v_k\}$ such that $v_{i_n} \xrightarrow{\text{desc}} v'_n$. By the assumptions of the Theorem there exist nodes w_{i_n} and w'_{i_n}, such that $w_{i_n} \xrightarrow{\text{desc}} w'_{i_n}$, $v'^{\circ}_n = w'^{\circ}_{i_n}$, $E_{v'_n} \subset E_{w'_{i_n}}$ and $|E_{w'_{i_n}}| \leq |E_{v'_n}| + \frac{2}{2^m}$. Now we let $v'_{n+1} := w'_{i_n}$. From this it immediately follows that

$$\frac{|E_{v'_n}|}{|E_{v'_{n+1}}|} \geq \frac{|E_{v'_n}|}{|E_{v'_n}| + \frac{2}{2^M}} \geq \frac{\delta}{\delta + \frac{2}{2^M}} = \frac{1}{1 + \frac{2}{\delta 2^M}}.$$

So, using that $\mu\left(E_{v'_n}\right) \leq \mu\left(E_{v'_{n+1}}\right)$ we obtain that

$$a_{v'_n} = \frac{|E_{v'_n}|^s}{\mu(E_{v'_n})} \geq \frac{|E_{v'_{n+1}}|^s}{\mu(E_{v'_{n+1}})(1 + \frac{2}{\delta 2^M})^s} = \frac{a_{v'_{n+1}}}{(1 + \frac{2}{\delta 2^M})^s}. \tag{5.7}$$

Note that for $n = 0, 1, 2, \ldots, l-1$ we have $E_{v_{i_n}} \subsetneq E_{w_{i_n}}$, $v_{i_n} \xrightarrow{\text{desc}} v'_n$, $w_{i_n} \xrightarrow{\text{desc}} v'_{n+1}$ and $E_{v'_1} \subsetneq E_{v'_2} \subsetneq \cdots \subsetneq E_{v'_l}$. This yields that $v_{i_0}, v_{i_1}, \ldots, v_{i_{l-1}}$ are all different. This follows that $l \leq k$ holds and $v'_l \in \mathcal{T}_D$. By applying (5.7) l times we get

$$a_{v'_0} \geq a_{v'_l} \Big/ \left(1 + \frac{2}{\delta 2^M}\right)^{l \cdot s} \geq B_D \Big/ \left(1 + \frac{2}{\delta 2^M}\right)^{k \cdot s} \geq \frac{B_D}{1 + \varepsilon},$$

which gives (5.6) and completes the proof. □

In the following we present the Algorithm. We remark that the starting set can be reduced by using symmetry. We will consider it at the end of this Section.

Algorithm 2. Step 1. *Let $Q_0 := C_0$ (which was defined in (3.10)).*

Step 2. *Let*

$$D_0 = \{v \,|\, v \in C_0, v \text{ is convex}\}.$$

Let $n := 0$.

Step 3. *Find $\min_{v \in D_n} b_v$. Below we prove that*

$$\min_{v \in D_n} b_v \leq \mathcal{H}^s(\Lambda) \tag{5.8}$$

holds.

Step 4. *Find a node $v \in D_n$ for which $b_v = \min_{w \in D_n} b_w$ (if such a v is not unique, then choose any of them). Let U_n be the set of non-convex descendants of v in one generation. That is*

$$U_n := \{w \,|\, w^\circ = v^\circ + 1,\, v \xrightarrow{desc} w, w \text{ is non-convex } \}.$$

$$V_n := \{w \,|\, w^\circ = v^\circ + 1,\, v \xrightarrow{desc} w,\, \exists \text{ a level } v^\circ + 1 \text{ triangle } \Delta \notin w,$$
$$\text{such that the conditions of Lemma 5 holds}$$
$$\text{by replacing } n \text{ with } w^\circ \text{ and } v \text{ with } w \text{ in Lemma } 5.\}$$

Moreover, we define

$$W_n := \{w \,|\, w^\circ = v^\circ + 1,\, v \xrightarrow{desc} w\} \setminus (U_n \cup V_n).$$

Note that the set $U_n \cup V_n \cup W_n$ contains all of those nodes which are descendants of the node v in one generation. Let

$$D_{n+1} := W_n \cup (D_n \setminus \{v\})\,.$$

Increase n by 1. Go to Step 3.

The only thing remained to be done is to verify (5.8). To do so, we will use Theorem 3. Let us fix n, and consider the set

$$Q_n = D_n \cup (C_0 \setminus D_0) \cup \bigcup_{k=0}^{n-1} U_k \cup V_k.$$

In the following we will check the assumptions of Theorem 3 by replacing Q with Q_n and D with D_n.

It is easy to see that Q_n is a cross-section, because

$$\mathcal{T}_{Q_n} \cup P_{Q_n} = \mathcal{T}_{C_0} \cup P_{C_0}.$$

For $v \in Q_n \setminus D_n$ there exists an $i = 0, 1, 2, \ldots, n-1$ such that an $v \in U_i \cup V_i$, or $v \in C_0 \setminus D_0$. If $v \in U_i$ or $v \in C_0 \setminus D_0$, then v is non-convex. Let $w = conv(v)$. We have $v^\circ = w^\circ$, and $E_v \subsetneq E_w$. Let $v \xrightarrow{\text{desc}} v'$ be arbitrary. By using Lemma 4 for v and $m = v'^\circ$, there exist w', $w'^\circ = v'^\circ = m$, such that

$$E_{v'} \subset E_{w'} \text{ and } |E_{w'}| \leq |E_{v'}| + \frac{2}{2^m}.$$

If $v \in V_i$, then the conditions of Lemma 5 holds for v, $n = v^\circ$ and for a level n triangle $\Delta \notin v$. Let $w = v \cup \{\Delta\}$. We have $v^\circ = w^\circ$, and $E_v \subsetneq E_w$. Let $v \xrightarrow{\text{desc}} v'$ be arbitrary. By using Lemma 5 there exists a level $m := v'^\circ$ triangle Δ', such that

$$|E_{v'} \cup \Delta'| \leq |E_{v'}| + \frac{1}{2^m}.$$

By choosing $w' = v' \cup \{\Delta'\}$, we obtain that

$$E_{v'} \subset E_{w'} \text{ and } |E_{w'}| \leq |E_{v'}| + \frac{2}{2^m}.$$

Using Theorem 3 we get

$$B_{D_n} \leq \mathcal{H}^s(\Lambda)$$

which completes the proof of (5.8).

By symmetry we can assume that for every level 4 descendants v of the node $\{E_1, E_2, E_3\}$ we have

$$\#(v \cap (\mathcal{T}_{\{E_{11}\}} \cup \mathcal{T}_{\{E_{12}\}} \cup \mathcal{T}_{\{E_{13}\}})) \leq \#(v \cap (\mathcal{T}_{\{E_{21}\}} \cup \mathcal{T}_{\{E_{22}\}} \cup \mathcal{T}_{\{E_{23}\}})) \leq$$
$$\leq \#(v \cap (\mathcal{T}_{\{E_{31}\}} \cup \mathcal{T}_{\{E_{32}\}} \cup \mathcal{T}_{\{E_{33}\}})).$$

To reduce the usage of the computer memory we modify the Algorithm 2. First we fix a constant Z. We store only those nodes, which are necessary to prove that a fixed constant Z is a lower bound on the Hausdorff measure of the Sierpinski triangle. Let

$$\overline{D}_n = \{v \,|\, v \in D_n, b_v \leq Z\}.$$

During the modified Algorithm we store $\overline{D}_n$ instead of D_n. We use D_n to find a node $v \in D_n$ such that $b_v = \min_{w \in D_n} b_w$. If $\overline{D}_n$ is the empty set, then

$$Z < \min_{v \in D_n} b_v \leq \mathcal{H}^s(\Lambda),$$

otherwise we have

$$\min_{v \in \overline{D}_n} b_v = \min_{v \in D_n} b_v.$$

If this modified Algorithm reaches a state where $\overline{D}_n = \emptyset$, then by using inequality (5.8) we have

$$Z < \mathcal{H}^s(\Lambda).$$

6. Running results

I wrote the program in *C++* language. For $Z = 0.73$ the program runs for half an hour, for $Z = 0.77$ it runs for a 4 days. The best result, what I managed to reach, is 0.77.

The program is available as an electric supplement at my homepage: *http://www.math.bme.hu/˜morap/sierpinski.zip*

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