BME Mathematics Contest **29 April 2019** 10^{15}–14^{00}, **Q-I**

— Corrected answer sheets are available for review to all participants in office H666 from 16:15 till 18:00 on 6 May. All interested are cordially invited to the ceremonial announcement of results and to the following reception at the cafeteria on the 4th floor of building Q from 6pm on 15 May.
— Each exercise is worth 10 points. Partial solutions are also considered. In some instances even more than 10 points are awarded on a problem (e.g. by providing interesting generalizations or multiple solutions).
— **Each problem should be answered on a separate sheet of paper** carrying the exercise number as well as the name and NEPTUN code of the participant.
— Morgan Stanley has offered extra prize for mathematical problems leading to ideas and methods which share some similarities to those one would use for financial and economical decision-making. This year the prize will be shared among the participants working out the best strategies of interception for exercise **4**.

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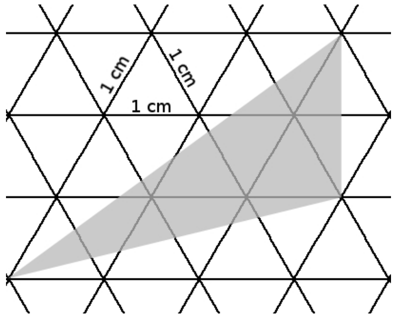
1. We have a paper with a triangle-grid on it composed of equilateral triangles with a side length of 1cm. We consider some triangles whose vertices lie only on grid intersection points (see the illustration showing an example for such a triangle), cut them out and successfully use them to completely cover (without overlap) all 6 faces of a rectangular box (a cuboid); each in a separate manner. Can the volume of the box be an integer in cubic centimeters?



2. $\lim_{n\to\infty} \sum_{k=n}^{2n} \binom{k-1}{n-1} 2^{-k} = ?$

3. If H is a set containing a given number $n > 1$ of (arbitrary) positive integers, how many elements can be in $\{xy + z | x, y, z \in H\}$ at most (2 points) and at least (8 points)?

4. On a field, there are n foxholes one after another, with an underground connection provided by a long, straight tunnel. The fox hides in one of the holes. The hunter pops the barrel of his shotgun into a hole and fires a round. If the poor creature was there, it dies. (*This is purely theoretical; no fox was hurt in the making of the problem.* ◡̈) If it wasn't there, then frightened by the gunshot, the fox moves by one hole using the underground tunnel; i.e. it moves into a hole adjacent to the one it was hiding before. The hunter can reload and fire without limits, each time choosing any of the holes he suspects the fox is currently hiding. Does there exist a strategy by which, after a certain number of shots, the hunter can be sure that the fox is dead? How many shots are needed for this?

5. Let $r > 0$ and $f : (0, r) \to \mathbb{R}$ be a continuous function with $\int_0^r |f(x)|dx < \infty$ and set $g(x) := \frac{1}{x}\int_0^x f(t)dt$ to be the average value of f on $[0, x]$. Prove that if $c = 4$, then $\int_0^r g(x)^2dx \le c\int_0^r f(x)^2dx$, and also that this is the smallest value for c for which this inequality always holds.

6. The complex $n \times n$ matrices A, B satisfy the relation $A^2B + BA^2 = 2ABA$. Check that $X = AB - BA$ commutes with A, and either using this or in any other way prove that $\exists k \in \{1, \dots n\} : X^k = 0$.

7. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Show that f is additive on orthogonal vectors – i.e. that $f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$ whenever $\mathbf{x}\cdot\mathbf{y} = \mathbf{0}$ – if and only if $\exists \mathbf{v} \in \mathbb{R}^n$ and $t \in \mathbb{R}$ such that $\forall \mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) = \mathbf{v}\cdot\mathbf{x} + t\,\mathbf{x}\cdot\mathbf{x}$.

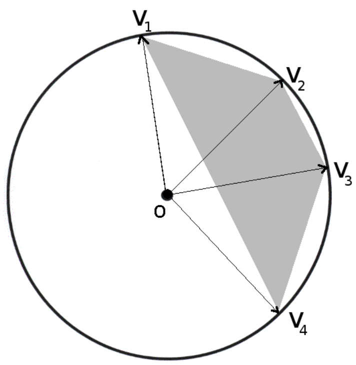
8. We shall say that a subset $H \subset \mathbb{Z}^+$ is *tasty*, if it satisfies the following:

i) any two (distinct) elements of H are relative primes,

ii) $\sum_{k\in S} k$ is a composite number whenever $\emptyset \neq S \subset H$.

Prove that there exist infinite tasty sets; either by showing that for an $H \subset \mathbb{Z}^+$ *finite* tasty set there always exists an $n \in \mathbb{Z}^+$ such that $H \cup \{n\}$ is still tasty, or in any other way!

9. Let $\mathbf{v}_1, \dots \mathbf{v}_n \in \mathbf{R}^d$ be a collection of unit vectors whose *convex hull* contains the origin (i.e. such that there exist some $t_1, \dots t_n \geq 0$ coefficients for which $\sum_j t_j = 1$ and $\sum_j t_j \mathbf{v}_j = \mathbf{0}$; see the illustration for a configuration *not* satisfying this condition). Prove that for any $k \in \mathbf{N}$, there exists a choice of indices $j_1, \dots j_k \in \{1, \dots n\}$ such that the length of the sum $\sum_{r=1}^k \mathbf{v}_{j_r}$ is at most $\sqrt{k}$. For bonus points, one can also show that this bound is *optimal*; that is, for any $\epsilon > 0$, there exist n, d and $\mathbf{v}_1, \dots \mathbf{v}_n \in \mathbf{R}^d$ unit vectors for which the length of $\sum_{r=1}^k \mathbf{v}_{j_r}$ — for any choice of $j_1, \dots j_k \in \{1, \dots n\}$ — is always greater than $\sqrt{k} - \epsilon$.



A *bad* configuration: $n = 4$ unit vectors in $d = 2$ dimension whose convex hull (the gray area) does *not* contain the origin.

10. Determine the best lower bound on the sum of squares

$$\left(\int_{-\infty}^{\infty} |f(x-1) + f(x+1)|^2 dx\right)^2 + \left(\int_{-\infty}^{\infty} |f(x-2) + f(x+2)|^2 dx\right)^2,$$

if the function $f : \mathbb{R} \to \mathbb{C}$ is such that $\int_{-\infty}^{\infty} |f(x)|^2 dx = 1$. *Computational help: the only real root of the polynomial* $2t + 4(2t-1)^3$ *is* $1/4$.