12. gyakorlat

Parciális törtekre bontás

1. a) $\int \frac{x+1}{x^2+3x}\,dx$ b)^{hf} $\int \frac{5x-3}{x^2-2x-3}\,dx$ c)^{hf} $\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx$

2. a) $\int \frac{1}{x^3+2x^2}\,dx$ b)^{hf} $\int \frac{x+1}{(x-1)^2(x-3)}\,dx$ c)^{hf} $\int \frac{x+5}{x^2+6x+9}\,dx$

3. a) $\int \frac{x+4}{x(x^2+2)}\,dx$ b)^{hf} $\int \frac{x^3}{x^4-16}\,dx$ c)^{hf} $\int \frac{x^5-15x}{x^4-16}\,dx$

Helyettesítéses integrálás

1. a) $\int x^2 \sin x^3\,dx$ b)^{hf} $\int \sin^4 x \cos dx$ c)^{hf} $\int \sin x\, e^{\cos x}\,dx$

2. a) $\int \frac{1}{\sqrt{x}+1}\,dx$ b) $\int \frac{9x}{\sqrt{2-3x}+1}\,dx$ c)^{hf} $\int x\sqrt{5x+3}\,dx$

3. a) $\int \frac{e^{2x}}{e^{2x}+1}\,dx$ b) $\int \frac{e^{6x}}{e^{2x}+1}\,dx$ c) $\int \frac{e^{2x}+5e^x}{e^{2x}+4e^x+3}\,dx$ d)^{hf} $\int \frac{e^x-1}{e^x+1}\,dx$

4.* a) $\int \frac{2x}{\sqrt{1-x^4}}\,dx$ b) $\int \frac{1}{x\sqrt{x^2-1}}\,dx$ c) $\int \sqrt{4-x^2}\,dx$ d) $\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

További gyakorló feladatok

1. a) $\int \frac{1}{x-\sqrt{x}}\,dx$ b) $\int \frac{\sqrt{x}}{1+x}\,dx$ c)* $\int \frac{1}{x^2}\sqrt[3]{\frac{x+1}{x}}\,dx$

2. a) $\int \sin\left(\sqrt{x}\right)dx$ b) $\int e^{\sqrt{x}}\,dx$ c) $\int (e^x)^2 \sin(e^x)\,dx$

3. a) $\int \frac{e^{2x}}{e^x+1}\,dx$ b)* $\int \frac{4}{e^{2x}-4}\,dx$ c)** $\int \frac{1}{x^3+1}\,dx$

Megoldások

Parciális törtekre bontás

1. a) $I = \int \frac{x+1}{x^2+3x}\,dx = ?$

Megoldás. A nevezőnek két különböző valós gyöke van: $x_1 = 0$, $x_2 = -3$.

Parciális törtekre bontás:

$$\frac{x+1}{x^2+3x} = \frac{x+1}{x(x+3)} = \frac{A}{x} + \frac{B}{x+3} = \frac{A(x+3)+Bx}{x(x+3)}$$

$\Longrightarrow\ x+1 = A(x+3) + B\,x$

Az A és B valós számokat úgy akarjuk meghatározni, hogy ez az egyenlőség **minden** x valós számra teljesüljön.

1. módszer (az együtthatók összehasonlítása)

$x+1 = (A+B)\,x + 3\,A$ pontosan akkor teljesül **minden** valós x-re, ha

$A+B=1 \Longrightarrow A = \frac{1}{3},\ B = \frac{2}{3}$

$3\,A = 1$

2. módszer (behelyettesítés) (célszerű a nevező gyökeit behelyettesíteni)

$x+1 = A(x+3) + B\,x$

ha $x=0 \Longrightarrow 1 = A\cdot 3 + B\cdot 0 \Longrightarrow A = \frac{1}{3}$

ha $x=-3 \Longrightarrow -2 = A\cdot 0 + B\cdot(-3) \Longrightarrow B = \frac{2}{3}$

$$\Longrightarrow I = \int \frac{x+1}{x^2+3\,x}\,dx = \int\left(\frac{1}{3}\cdot\frac{1}{x} + \frac{2}{3}\cdot\frac{1}{x+3}\right)dx = \frac{1}{3}\ln|x| + \frac{2}{3}\ln|x+3| + c$$

wolframalpha.com: integrate (x+1)/(x^2+3x)

https://www.wolframalpha.com/input?i=integrate+%28x%2B1%29%2F%28x%5E2%2B3x%29

1. b) $I = \int \frac{5\,x-3}{x^2-2\,x-3}\,dx = ?$

Hasonlóan: $\frac{5\,x-3}{x^2-2\,x-3} = \frac{5\,x-3}{(x-3)(x+1)} = \frac{A}{x+1} + \frac{B}{x-3} = \ldots = \frac{2}{x+1} + \frac{3}{x-3}$

(az A és B együtthatót határozzuk meg kétféleképpen!)

$$\Longrightarrow I = \int\left(\frac{2}{x+1} + \frac{3}{x-3}\right)dx = 2\ln|x+1| + 3\ln|x-3| + c$$

wolframalpha.com: integrate (5x-3)/(x^2-2x-3)

https://www.wolframalpha.com/input?i=integrate+%285x-3%29%2F%28x%5E2-2x-3%29

1. c) $I = \int \frac{2\,x^3-4\,x^2-x-3}{x^2-2\,x-3}\,dx = ?$

Polinomosztással: $\frac{2\,x^3-4\,x^2-x-3}{x^2-2\,x-3} = 2\,x + \frac{5\,x-3}{x^2-2\,x-3}$,

ahol a tört ugyanaz, mint az 1. b) feladatban

$$\Longrightarrow I = \int\left(2\,x + \frac{2}{x+1} + \frac{3}{x-3}\right)dx = x^2 + 2\ln|x+1| + 3\ln|x-3| + c$$

2. a) $\int \frac{1}{x^3+2\,x^2}\,dx$ **b)** $\int \frac{x+1}{(x-1)^2\,(x-3)}\,dx$

Példatár: 5. fejezet, 7., 8. feladat
https://math.bme.hu/~tasnadi/merninf_anal_1/anal1_gyak.pdf

wolframalpha.com: integrate 1/(x^3+2x^2)
https://www.wolframalpha.com/input?i=integrate+1%2F%28x%5E3%2B2x%5E2%29

2. c) $I = \int \frac{x+5}{x^2+6x+9}\,dx = ?$

Megoldás. A nevezőnek kétszeres valós gyöke van: $x_{1,2} = -3$.

Parciális törtekre bontás:

$$\frac{x+5}{x^2+6x+9} = \frac{x+5}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2} = \frac{A(x+3)+B}{(x+3)^2}$$

$$\Longrightarrow\; x+5 = A(x+3)+B$$

1. módszer: $x+5 = A\,x + (3\,A+B) \;\Longrightarrow\; \begin{matrix} A = 1 \\ 3\,A+B=5 \end{matrix} \;\Longrightarrow\; A=1,\; B=2$

2. módszer: $x+5 = A(x+3)+B$

$x=-3 \;\Longrightarrow\; 2 = 0+B \;\Longrightarrow B=2$

$x=0 \;\Longrightarrow\; 5 = 3\,A+B \;\Longrightarrow\; A=1$

$$\Longrightarrow\; I = \int \frac{x+5}{x^2+6x+9}\,dx = \int\left(\frac{1}{x+3} + \frac{2}{(x+3)^2}\right)dx = \int\left(\frac{1}{x+3} + 2\,(x+3)^{-2}\right)dx =$$

$$= \ln\left|\,x+3\,\right| + 2\,\frac{(x+3)^{-1}}{-1} + c = \ln\left|\,x+3\,\right| - \frac{2}{x+3} + c$$

3. a) $I = \int \frac{x+4}{x(x^2+2)}\,dx = ?$

Megoldás. A nevező gyökei:

$x_1 = 0$ (egyszeres valós gyök), $x_{2,3} = \pm 2\,i$ (egyszeres komplex gyökök)

Parciális törtekre bontás:

$$\frac{x+4}{x(x^2+2)} = \frac{A}{x} + \frac{B\,x+C}{x^2+2} = \frac{A(x^2+2)+(B\,x+C)\cdot x}{x(x^2+2)}$$

$$\Longrightarrow\; x+4 = A(x^2+2)+(B\,x+C)\cdot x$$

Helyettesítések (behelyettesítsjük a nevező valós gyökét, továbbá még két tetszőleges valós számot):

$x=0 \;\Longrightarrow\; 4 = 2\,A+0 \;\Longrightarrow\; A=2$

$x=1 \;\Longrightarrow\; 5 = 3\,A+B+C$

$x=-1 \;\Longrightarrow\; 3 = 3\,A+B-C \;\Longrightarrow\; C=1,\; B=-2$

$$\Longrightarrow\; I = \int \frac{x+4}{x(x^2+2)}\,dx = = \int\left(\frac{2}{x} + \frac{-2\,x+1}{x^2+2}\right)dx =$$

Átalakítás: $\frac{-2x+1}{x^2+2} = -\frac{2x}{x^2+2} + \frac{1}{x^2+2} = -\frac{2x}{x^2+2} + \frac{1}{2}\frac{1}{\left(\frac{x}{\sqrt{2}}\right)^2+1}$

Felhasználjuk:

- $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$
- $\int \frac{1}{x^2+1}\,dx = \operatorname{arctg} x + c, \qquad \int f(ax+b)\,dx = \frac{F(ax+b)}{a} + c$, ahol $F' = f,\ a \neq 0$

$$\Longrightarrow\ I = \int\left(\frac{2}{x} - \frac{2x}{x^2+2} + \frac{1}{2}\frac{1}{\left(\frac{x}{\sqrt{2}}\right)^2+1}\right)dx = 2\ln|x| - \ln(x^2+2) + \frac{1}{2}\cdot\frac{\operatorname{arctg}\left(\frac{x}{\sqrt{2}}\right)}{\frac{1}{\sqrt{2}}} + c$$

wolframalpha.com: integrate (x+4)/(x(x^2+2))
https://www.wolframalpha.com/input?i=integrate+%28x%2B4%29%2F%28x%28x%5E2%2B2%29%29%2
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3. b) $\int \frac{x^3}{x^4-16}\,dx$ **c)** $\int \frac{x^5-15x}{x^4-16}\,dx$

Példatár: 5.3 fejezet, 9. feladat
https://math.bme.hu/~tasnadi/merninf_anal_1/anal1_gyak.pdf

Helyettesítéses integrálás

1. a) $I = \int x^2 \sin x^3\,dx = ?$

Helyettesítés: $t = x^3 \Longrightarrow x = x(t) = \sqrt[3]{t} = t^{\frac{1}{3}} \Longrightarrow x^2 = t^{\frac{2}{3}}$

$$x'(t) = \frac{dx}{dt} = \frac{1}{3}t^{-\frac{2}{3}} \Longrightarrow dx = \frac{1}{3}t^{-\frac{2}{3}}\,dt$$

$$I = \int x^2 \sin(x^3)\,dx = \int t^{\frac{2}{3}} \sin t \cdot \frac{1}{3}t^{-\frac{2}{3}}\,dt = \int \frac{1}{3}\sin t\,dt = -\frac{1}{3}\cos t + c = -\frac{1}{3}\cos x^3 + c$$

1. b) $I = \int \sin^4 x \cos dx = ?$

1. megoldás. $\int f'(x) f^{\alpha}(x)\,dx = \frac{f^{\alpha+1}(x)}{\alpha+1} + c \quad (\alpha \neq -1)$

Here: $f(x) = \sin x,\ f'(x) = \cos x,\ \alpha = 4 \Longrightarrow$

$$I = \int \sin^4 x \cos x\,dx = \frac{\sin^5 x}{5} + c$$

2. megoldás

Helyettesítés: $t = \sin x \Longrightarrow x = x(t) = \arcsin t$

$$x'(t) = \frac{dx}{dt} = \frac{1}{\sqrt{1-t^2}} \Longrightarrow dx = \frac{1}{\sqrt{1-t^2}}\,dt$$

$$\cos x = \sqrt{\cos^2 x} = \sqrt{1-\sin^2 x} = \sqrt{1-t^2}$$

$$I = \int \sin^4 x \cos x \, dx = \int t^4 \sqrt{1-t^2} \cdot \frac{1}{\sqrt{1-t^2}} \, dt = \int t^4 \, dt = \frac{t^5}{5} + c = \frac{\sin^5 x}{5} + c$$

1. c) $I = \int \sin x \, e^{\cos x} \, dx = ?$

Helyettesítés: $t = \cos x \implies x = x(t) = \arccos t$

$$x'(t) = \frac{dx}{dt} = -\frac{1}{\sqrt{1-t^2}} \implies dx = -\frac{1}{\sqrt{1-t^2}} \, dt$$

$$\sin x = \sqrt{\sin^2 x} = \sqrt{1-\cos^2 x} = \sqrt{1-t^2}$$

$$I = \int \sin x \, e^{\cos x} \, dx = \int \sqrt{1-t^2} \, e^t \cdot \left(-\frac{1}{\sqrt{1-t^2}}\right) dt = \int -e^t \, dt = -e^t + c = -e^{\cos x} + c$$

2. a) $I = \int \frac{1}{\sqrt{x}+1} \, dx = ?$

Helyettesítés: $t = \sqrt{x} \implies x = x(t) = t^2$

$$x'(t) = \frac{dx}{dt} = 2t \implies dx = 2t \, dt$$

$$I = \int \frac{1}{\sqrt{x}+1} \, dx = \int \frac{2t}{t+1} \, dt = \int \frac{2(t+1)-1}{t+1} \, dt = \int 2 \cdot \left(1 - \frac{1}{t+1}\right) dt =$$

$$= 2t - 2\ln|1+t| + c = 2\sqrt{x} - 2\ln\left(1+\sqrt{x}\right) + c$$

2. b) $I = \int \frac{9x}{\sqrt{2-3x}+1} \, dx = ?$

Példatár: 5. fejezet, 25. feladat

https://math.bme.hu/~tasnadi/merninf_anal_1/anal1_gyak.pdf

2. c) $I = \int x \sqrt{5x+3} \, dx = ?$

Helyettesítés: $t = \sqrt{5x+3} \implies x = x(t) = \frac{t^2-3}{5}$

$$x'(t) = \frac{dx}{dt} = \frac{2}{5} t \implies dx = \frac{2}{5} t \, dt$$

$$I = \int x \sqrt{5x+3} \, dx = \int \frac{t^2-3}{5} \cdot t \cdot \frac{2}{5} t \, dt = \int \frac{2}{25} (t^4 - 3t^2) \, dt = \frac{2}{25} \left(\frac{t^5}{5} - t^3\right) + c =$$

$$= \frac{2}{25} \left(\frac{\sqrt{(5x+3)^5}}{5} - \sqrt{(5x+3)^3}\right) + c$$

3. a) $I = \int \frac{e^{2x}}{e^{2x}+1} \, dx = ?$ **b)** $I = \int \frac{e^{6x}}{e^{2x}+1} \, dx = ?$

3. a), b): Példatár: 5.7 fejezet, 24. feladat

https://math.bme.hu/~tasnadi/merninf_anal_1/anal1_gyak.pdf

3. c) $I = \int \frac{e^{2x} + 5e^x}{e^{2x} + 4e^x + 3}\,dx = ?$

Helyettesítés: $e^x = t \implies x = x(t) = \ln t$

$$x'(t) = \frac{dx}{dt} = \frac{1}{t} \implies dx = \frac{1}{t}\,dt$$

$$I = \int \frac{e^{2x} + 5e^x}{e^{2x} + 4e^x + 3}\,dx = \int \frac{t^2 + 5e^t}{t^2 + 4t + 3} \cdot \frac{1}{t}\,dt = \int \frac{t+5}{(t+1)(t+3)}\,dt = \ldots = \int \left(\frac{2}{t+1} - \frac{1}{t+3}\right) dt =$$

$$= 2\ln|t+1| - \ln|t+3| + c = 2\ln(e^x + 1) - \ln(e^x + 3) + c$$

3. d) $I = \int \frac{e^x - 1}{e^x + 1}\,dx = ?$

Helyettesítés: $e^x = t \implies x = x(t) = \ln t$

$$x'(t) = \frac{dx}{dt} = \frac{1}{t} \implies dx = \frac{1}{t}\,dt$$

$$I = \int \frac{e^x - 1}{e^x + 1}\,dx = \int \frac{t-1}{t+1} \cdot \frac{1}{t}\,dt = \int \frac{t-1}{t(t+1)}\,dt = \ldots = \int \left(\frac{2}{t+1} - \frac{1}{t}\right) dt =$$

$$= 2\ln|t+1| - \ln|t| + c = 2\ln(e^x + 1) - \ln(e^x) + c = 2\ln(e^x + 1) - x + c$$

4. a) $I = \int \frac{2x}{\sqrt{1 - x^4}}\,dx = ?$

Helyettesítés: $t = x^2 \implies x = x(t) = \sqrt{t} = t^{\frac{1}{2}}$

$$x'(t) = \frac{dx}{dt} = \frac{1}{2} t^{-\frac{1}{2}} = \frac{1}{2\sqrt{t}} \implies dx = \frac{1}{2\sqrt{t}}\,dt$$

$$I = \int \frac{2x}{\sqrt{1 - x^4}}\,dx = \int \frac{2\sqrt{t}}{\sqrt{1 - t^2}} \cdot \frac{1}{2\sqrt{t}}\,dt = \int \frac{1}{\sqrt{1 - t^2}}\,dt = \arcsin t + c = \arcsin\left(x^2\right) + c$$

4. b) $I = \int \frac{1}{x\sqrt{x^2 - 1}}\,dx = ?$

Helyettesítés: $t = \sqrt{x^2 - 1} \implies t^2 = x^2 - 1 \implies t^2 + 1 = x^2$

$$x = x(t) = \sqrt{t^2 + 1} = \left(t^2 + 1\right)^{\frac{1}{2}}$$

$$x'(t) = \frac{dx}{dt} = \frac{1}{2}\left(t^2 + 1\right)^{-\frac{1}{2}} \cdot 2t = \frac{t}{\sqrt{t^2 + 1}} \implies dx = \frac{t}{\sqrt{t^2 + 1}}\,dt$$

$$I = \int \frac{1}{x\sqrt{x^2 - 1}}\,dx = \int \frac{1}{\sqrt{t^2 + 1} \cdot t} \cdot \frac{t}{\sqrt{t^2 + 1}}\,dt = \int \frac{1}{t^2 + 1}\,dt =$$

$$= \operatorname{arctg} t + c = \operatorname{arctg}\left(\sqrt{x^2 - 1}\right) + c$$

4. c) $I = \int \sqrt{4 - x^2}\,dx = ?$

Helyettesítés: $x = x(t) = 2\sin t \implies t = \arcsin\left(\frac{x}{2}\right)$

$$x'(t) = \frac{dx}{dt} = 2\cos t \implies dx = 2\cos t\, dt$$

$$I = \int \sqrt{4 - x^2}\, dx = \int \sqrt{4 - 4\sin^2 t} \cdot 2\cos t\, dt = \int 2\sqrt{1 - \sin^2 t} \cdot 2\cos t\, dt = \int 4\sqrt{\cos^2 t} \cdot \cos t\, dt$$

$$= \int 4\cos^2 t\, dt = \int 4 \cdot \frac{1 + \cos 2t}{2}\, dt = \int 2 \cdot (1 + \cos 2t)\, dt = 2\left(t + \frac{\sin 2t}{2}\right) + c =$$

$$= 2t + \sin 2t + c = 2 \cdot \arcsin\left(\frac{x}{2}\right) + 2 \cdot \frac{x}{2}\sqrt{1 - \left(\frac{x}{2}\right)^2} + c$$

Azonosságok:

$\cos^2 x + \sin^2 x = 1 \quad \implies \cos^2 x = \frac{1 + \cos 2x}{2}, \quad \sin^2 x = \frac{1 - \cos 2x}{2}$

$\cos^2 x - \sin^2 x = \cos 2x$

$\sin 2t = 2\sin t\cos t = 2\sin t\sqrt{\cos^2 t} = 2\sin t\sqrt{1 - \sin^2 t} = 2 \cdot \frac{x}{2} \cdot \sqrt{1 - \left(\frac{x}{2}\right)^2}$

4. d) $I = \int \frac{x^2}{\sqrt{9 - x^2}}\, dx = ?$

Helyettesítés: $x = 3\sin t \implies t = \arcsin\left(\frac{x}{3}\right)$

$$x'(t) = \frac{dx}{dt} = 3\cos t \implies dx = 3\cos t\, dt$$

$$I = \int \frac{x^2}{\sqrt{9 - x^2}}\, dx = \int \frac{(3\sin t)^2}{\sqrt{9 - (3\sin t)^2}} \cdot 3\cos t\, dt = \int \frac{9\sin^2 t}{\sqrt{9 \cdot (1 - \sin^2 t)}} \cdot 3\cos t\, dt =$$

$$= \int \frac{9\sin^2 t}{\sqrt{\cos^2 t}} \cos t\, dt = \int \frac{9\sin^2 t}{\cos t} \cos t\, dt = \int 9\sin^2 t\, dt = \int 9 \cdot \frac{1 - \cos 2t}{2}\, dt =$$

$$= \frac{9}{2} \cdot \left(t - \frac{\sin 2t}{2}\right) + c = \frac{9}{2} t - \frac{9}{4} \sin 2t + c = \frac{9}{2} \arcsin\left(\frac{x}{3}\right) - \frac{9}{4} \cdot 2 \cdot \frac{x}{3}\sqrt{1 - \left(\frac{x}{3}\right)^2} + c =$$

Azonosságok:

$\sin^2 x = \frac{1 - \cos 2x}{2}$

$\sin 2t = 2\sin t\cos t = 2\sin t\sqrt{\cos^2 t} = 2\sin t\sqrt{1 - \sin^2 t} = 2 \cdot \frac{x}{3} \cdot \sqrt{1 - \left(\frac{x}{3}\right)^2}$

További gyakorló feladatok

1. a) $I = \int \frac{1}{x - \sqrt{x}}\, dx = ?$

Helyettesítés: $t = \sqrt{x} \implies x = x(t) = t^2$

$$x'(t) = \frac{dx}{dt} = 2t \implies dx = 2t\, dt$$

$$I=\int\frac{1}{x-\sqrt{x}}\,\mathrm{dx}=\int\frac{1}{t^2-t}\cdot 2\,t\,\mathrm{dt}=\int\frac{2}{t-1}\,\mathrm{dt}=2\cdot\ln|\,t-1\,|+c=2\ln\left|\sqrt{x}-1\right|+c$$

1. b) $I=\int\frac{\sqrt{x}}{1+x}\,\mathrm{dx}=\,?$

Helyettesítés: $t=\sqrt{x}\implies x=x(t)=t^2$

$$x'(t)=\frac{\mathrm{dx}}{\mathrm{dt}}=2\,t\implies \mathrm{dx}=2\,t\,\mathrm{dt}$$

$$I=\int\frac{\sqrt{x}}{1+x}\,\mathrm{dx}=\int\frac{t}{1+t^2}\cdot 2\,t\,\mathrm{dt}=\int\frac{2\,t^2}{1+t^2}\,\mathrm{dt}=\int\frac{(2\,t^2+2)-2}{1+t^2}\,\mathrm{dt}=\int\left(2-\frac{2}{1+t^2}\right)\mathrm{dt}=$$
$$=2\,t-2\,\mathrm{arctg}\,t+c=2\sqrt{x}-2\,\mathrm{arctg}\sqrt{x}+c$$

1. c) $I=\int\frac{1}{x^2}\sqrt[3]{\frac{x+1}{x}}\,\mathrm{dx}=\,?$

Helyettesítés: $t=\sqrt[3]{\frac{x+1}{x}}\implies x+1=t^3\,x$

$$x=x(t)=\frac{1}{t^3-1}$$
$$x'(t)=\frac{\mathrm{dx}}{\mathrm{dt}}=-\frac{3\,t^2}{(t^3-1)^2}\implies \mathrm{dx}=-\frac{3\,t^2}{(t^3-1)^2}\,\mathrm{dt}$$

$$I=\int\frac{1}{x^2}\sqrt[3]{\frac{x+1}{x}}\,\mathrm{dx}=\int(t^3-1)^2\cdot t\cdot\frac{-3\,t^2}{(t^3-1)^2}\,\mathrm{dt}=\int-3\,t^3\,\mathrm{dt}=-\frac{3}{4}\,t^4+c=-\frac{3}{4}\left(\frac{x+1}{x}\right)^{\frac{4}{3}}+c$$

2. a) $I=\int\sin\left(\sqrt{x}\right)\mathrm{dx}=\,?$

Helyettesítés: $t=\sqrt{x}\implies x=x(t)=t^2$

$$x'(t)=\frac{\mathrm{dx}}{\mathrm{dt}}=2\,t\implies \mathrm{dx}=2\,t\,\mathrm{dt}$$

$$I=\int\sin\left(\sqrt{x}\right)\mathrm{dx}=\int 2\,t\sin t\,\mathrm{dt}=2\,t\cdot(-\cos t)-\int 2\cdot(-\cos t)\,\mathrm{dt}=-2\,t\cos t+2\sin t+c=$$
$$=-2\sqrt{x}\cos\left(\sqrt{x}\right)+2\sin\left(\sqrt{x}\right)+c\quad\text{(parciális integrálással)}$$

2. b) $I=\int e^{\sqrt{x}}\,\mathrm{dx}=\,?$

Helyettesítés: $t=\sqrt{x}\implies x=x(t)=t^2$

$$x'(t)=\frac{\mathrm{dx}}{\mathrm{dt}}=2\,t\implies \mathrm{dx}=2\,t\,\mathrm{dt}$$

$$I=\int e^{\sqrt{x}}\,\mathrm{dx}=\int 2\,t\,e^t\,\mathrm{dt}=2\,t\cdot e^t-\int 2\cdot e^t\,\mathrm{dt}=2\,t\cdot e^t-2\,e^t+c=$$
$$=2\sqrt{x}\cdot e^{\sqrt{x}}-2\,e^{\sqrt{x}}+c\quad\text{(parciális integrálással)}$$

3. c) $I=\int(e^x)^2\sin(e^x)\,\mathrm{dx}=\,?$

Helyettesítés: $e^x=t\implies x=x(t)=\ln t$

$$x'(t)=\frac{\mathrm{dx}}{\mathrm{dt}}=\frac{1}{t}\implies \mathrm{dx}=\frac{1}{t}\,\mathrm{dt}$$

$$I = \int (e^x)^2 \sin(e^x)\, dx = \int t^2 \cdot \sin t \cdot \frac{1}{t}\, dt = \int t \cdot \sin t\, dt = t \cdot (-\cos t) - \int 1 \cdot (-\cos t)\, dt =$$
$$= -t \cos t + \sin t + c = -e^x \cos(e^x) + \sin(e^x) + c \quad \text{(parciális integrálással)}$$

3. a) $I = \int \frac{e^{2x}}{e^x + 1}\, dx = ?$

Helyettesítés: $t = e^x \Longrightarrow x = x(t) = \ln t$

$$x'(t) = \frac{dx}{dt} = \frac{1}{t} \Longrightarrow dx = \frac{1}{t}\, dt$$

$$I = \int \frac{e^{2x}}{e^x + 1}\, dx = \int \frac{t^2}{t+1} \cdot \frac{1}{t}\, dt = \int \frac{t}{t+1}\, dt = \int \frac{(t+1) - 1}{t+1}\, dt = \int \left(1 - \frac{1}{t+1}\right) dt =$$
$$= t - \ln|t+1| + c = e^x - \ln(e^x + 1) + c$$

3. b) $I = \int \frac{4}{e^{2x} - 4}\, dx = ?$

Helyettesítés: $t = e^x \Longrightarrow x = x(t) = \ln t$

$$x'(t) = \frac{dx}{dt} = \frac{1}{t} \Longrightarrow dx = \frac{1}{t}\, dt$$

$$I = \int \frac{4}{e^{2x} - 4}\, dx = \int \frac{4}{t^2 - 4} \cdot \frac{1}{t}\, dt = \int \frac{4}{t(t-2)(t+2)}\, dt$$

Parciális törtekre bontás:

$$\frac{4}{t(t-2)(t+2)} = \frac{A}{t} + \frac{B}{t+2} + \frac{C}{t-2}$$

$\Longrightarrow \quad 4 = A(t+2)(t-2) + B\,t(t-2) + C\,t(t+2)$

$t = 0$: $\quad 4 = -4A + 0 + 0 \Longrightarrow A = -1$

$t = -2$: $\quad 4 = 0 + 8B + 0 \Longrightarrow B = \frac{1}{2}$

$t = 2$: $\quad 4 = 0 + 0 + 8C \Longrightarrow C = \frac{1}{2}$

$$\Longrightarrow I = \int \left(-\frac{1}{t} + \frac{1}{2}\frac{1}{t+2} + \frac{1}{2}\frac{1}{t-2}\right) dt = -\ln|t| + \frac{1}{2}\ln|t+2| + \frac{1}{2}\ln|t-2| + c =$$
$$= -\ln e^x + \frac{1}{2}\ln(e^x + 2) + \frac{1}{2}\ln|e^x - 2| + c$$

3. c) $I = \int \frac{1}{x^3 + 1}\, dx = ?$

Parciális törtekre bontva: $\frac{1}{x^3 + 1} = \frac{1}{(x+1)(x^2 - x + 1)} = \frac{1}{3}\frac{1}{x+1} - \frac{1}{3}\frac{x-2}{x^2 - x + 1}$

Az integrál: $I = \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln|x^2 - x + 1| + \frac{1}{\sqrt{3}}\,\mathrm{arctg}\left(\frac{2x-1}{\sqrt{3}}\right)$