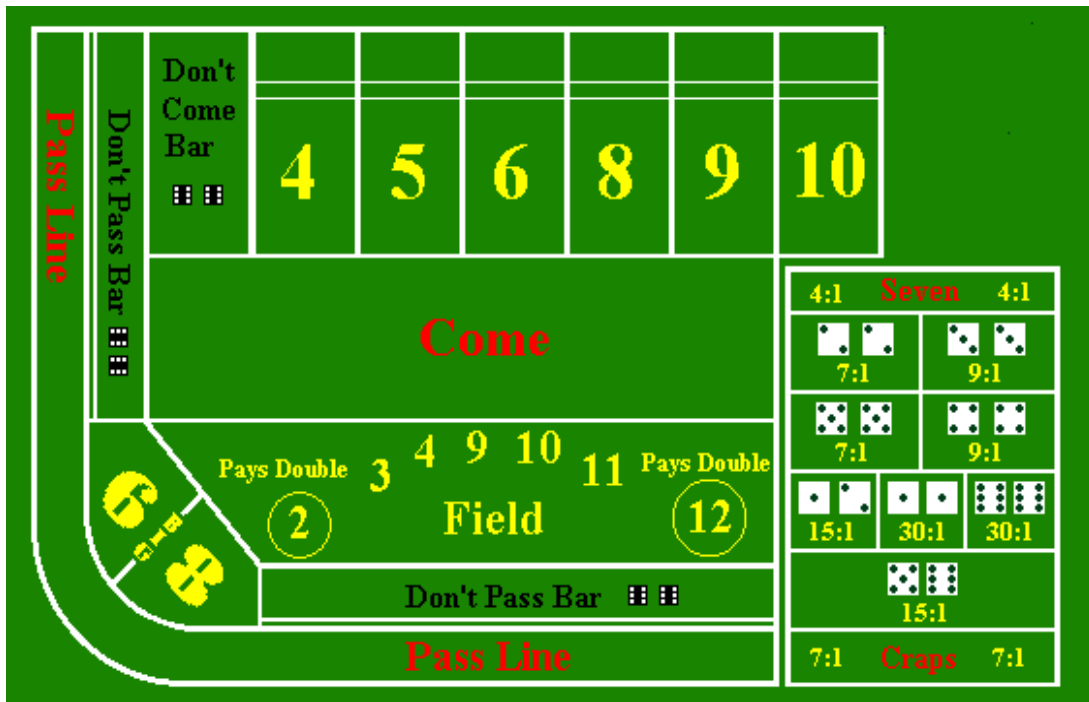


## 4. Craps

**Craps** is a popular casino game, because of its complexity and because of the rich variety of bets that can be made. A typical craps table is shown below:



According to **Richard Epstein**, craps is descended from an earlier game known as **Hazard**, that dates to the Middle Ages. The formal rules for Hazard were established by **Montmort** early in the 1700s. The origin of the name *craps* is shrouded in doubt, but it may have come from the English *crabs* or from the French *Crapeaud* (for toad).

From a mathematical point of view, craps is interesting because it is an example of a **random experiment** that takes place in stages; the evolution of the game depends critically on the outcome of the first roll. In particular, the number of rolls is a **random variable**.

### Definition of the Game

The rules for craps are as follows: The player (known as the **shooter**) rolls a pair of fair dice

1. If the sum is 7 or 11 on the first throw, the shooter wins; this event is called a **natural**.
2. If the sum is 2, 3, or 12 on the first throw, the shooter loses; this event is called **craps**.
3. If the sum is 4, 5, 6, 8, 9, or 10 on the first throw, this number becomes the shooter's **point**. The shooter continues rolling the dice until either she rolls the point again (in which case she wins) or rolls a 7 (in which case she loses).

As long as the shooter wins, or loses by rolling craps, she retains the dice and continues. Once she loses by failing to

make her point, the dice are passed to the next shooter.

Let us consider the game of craps mathematically. Our basic assumption, of course, is that the dice are fair and that the outcomes of the various rolls are **independent**. Let  $N$  denote the (random) number of rolls in the game and let  $(X_i, Y_i)$  denote the outcome of the  $i^{\text{th}}$  roll for  $i \in \{1, 2, \dots, N\}$ . Finally, let  $Z_i = X_i + Y_i$ , the sum of the scores on the  $i^{\text{th}}$  roll, and let  $I$  denote the indicator variable that the player wins.

1. In the **craps experiment**, press single step a few times and observe the outcomes. Make sure that you understand the rules of the game.

## The Probability of Winning

We will compute the probability that the shooter wins in stages, based on the outcome of the first roll.

2. Show that  $Z_i$  has the probability density function given in the following table:

$z$	2	3	4	5	6	7	8	9	10	11	12
$\mathbb{P}(Z_1 = z)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

The probability that the player makes her point can be computed using a simple conditioning argument. For example, suppose that the player throws 4 initially, so that 4 is the point. The player continues until she either throws 4 again or throws 7. Thus, the final roll will be an element of the following set:

$$S_4 = \{(1, 3), (2, 2), (3, 1), (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$$

Since the dice are fair, these outcomes are equally likely, so the probability that the player makes her 4 point is  $\frac{3}{9}$ . A similar argument can be used for the other points. The results are summarized in the following exercise.

3. Show that the probability of making the point  $z$  are as given in the following table:

$z$	4	5	6	8	9	10
$\mathbb{P}(I = 1   Z_1 = z)$	$\frac{3}{9}$	$\frac{4}{10}$	$\frac{5}{11}$	$\frac{5}{11}$	$\frac{4}{10}$	$\frac{3}{9}$

4. Use the result of Exercises 1 and 3 to show that

$$\mathbb{P}(I = 1) = \frac{244}{495} \approx 0.49292, \quad \mathbb{P}(I = 0) = \frac{251}{495} \approx 0.50707$$

Note that craps is nearly a fair game.

## Bets

There are a bewildering variety of bets that can be made in craps. In the exercises in this subsection, we will discuss some typical bets and compute their **density**, **mean**, and **standard deviation**. (Most of these bets are illustrated in the picture of the craps table above). Note however, that some of the details of the bets and, in particular the payout odds, vary from one casino to another. Of course the expected value of any bet is inevitably negative (for the gambler), and thus the gambler is

doomed to lose money in the long run. Nonetheless, as we will see, some bets are better than others.

### Pass and Don't Pass

A **pass bet** is a bet that the shooter will win and pays 1 : 1.

5. Let  $W$  denote the winnings from a unit pass bet. Show that

- $\mathbb{P}(W = -1) = \frac{251}{495}$ ,  $\mathbb{P}(W = 1) = \frac{244}{495}$
- $\mathbb{E}(W) \approx -0.0141$
- $\text{sd}(W) \approx 0.9999$

6. In the **craps experiment**, select the pass bet. Run the simulation 1000 times, updating every 10 runs and watch the apparent convergence of the empirical density and moments of  $W$  to the true density and moments. Suppose that you bet \$1 on each of the 1000 games. What would your net winnings be?

A **don't pass bet** is a bet that the shooter will lose, except that 12 on the first throw is excluded (that is, the shooter loses, of course, but the don't pass bet neither wins nor loses). This is the meaning of the phrase *don't pass bar double 6* on the craps table. The don't pass bet also pays 1 : 1.

7. Let  $W$  denote the winnings for a unit don't pass bet. Show that

- $\mathbb{P}(W = -1) = \frac{244}{495}$ ,  $\mathbb{P}(W = 0) = \frac{1}{36}$ ,  $\mathbb{P}(W = 1) = \frac{949}{1980}$
- $\mathbb{E}(W) \approx -0.01363$
- $\text{sd}(W) \approx 0.9859$

8. In the **craps experiment**, select the don't pass bet. Run the simulation 1000 times, updating every 10 runs and watch the apparent convergence of the empirical density and moments of  $W$  to the true density and moments. Suppose that you bet \$1 on each of the 1000 games. What would your net winnings be?

The **come bet** and the **don't come bet** are analogous to the pass and don't pass bets, respectively, except that they are made *after* the point has been established.

### Field

A **field bet** is a bet on the outcome of the next throw. It pays 1 : 1 if 3, 4, 9, 10, or 11 is thrown, 2 : 1 if 2 or 12 is thrown, and loses otherwise.

9. Let  $W$  denote the winnings for a unit field bet. Show that

- $\mathbb{P}(W = -1) = \frac{5}{9}$ ,  $\mathbb{P}(W = 1) = \frac{7}{18}$ ,  $\mathbb{P}(W = 2) = \frac{1}{18}$
- $\mathbb{E}(W) \approx -0.0556$
- $\text{sd}(W) \approx 1.0787$

10. In the **craps experiment**, select the field bet. Run the simulation 1000 times, updating every 10 runs and watch the apparent convergence of the empirical density and moments of  $W$  to the true density and moments. Suppose that you bet \$1 on each of the 1000 games. What would your net winnings be?

### Seven and Eleven

A **7 bet** is a bet on the outcome of the next throw. It pays 4 : 1 if a 7 is thrown and loses otherwise. Similarly, an **11 bet** is a bet on the outcome of the next throw, and pays 15 : 1 if an 11 is thrown. In spite of the romance of the number 7, the next exercise shows that the 7 bet is one of the worst bets you can make.

11. Let  $W$  denote the winnings for a unit 7 bet. Show that

- $\mathbb{P}(W = -1) = \frac{5}{6}, \mathbb{P}(W = 4) = \frac{1}{6},$
- $\mathbb{E}(W) \approx -0.1667$
- $\text{sd}(W) \approx 1.8634$

12. In the **craps experiment**, select the 7 bet. Run the simulation 1000 times, updating every 10 runs and watch the apparent convergence of the empirical density and moments of  $W$  to the true density and moments. Suppose that you bet \$1 on each of the 1000 games. What would your net winnings be?

13. Let  $W$  denote the winnings for a unit 11 bet. Show that

- $\mathbb{P}(W = -1) = \frac{17}{18}, \mathbb{P}(W = 15) = \frac{1}{18}.$
- $\mathbb{E}(W) \approx -0.1111$
- $\text{sd}(W) \approx 3.6650$

14. In the **craps experiment**, select the 11 bet. Run the simulation 1000 times, updating every 10 runs and watch the apparent convergence of the empirical density and moments of  $W$  to the true density and moments. Suppose that you bet \$1 on each of the 1000 games. What would your net winnings be?

### Craps

All **craps bets** are bets on the next throw. The basic **craps bet** pays 7 : 1 if 2, 3, or 12 is thrown, and loses otherwise. The **craps 2 bet** pays 30 : 1 if a 2 is thrown and loses otherwise. Similarly, the craps 12 bet pays 30 : 1 if a 12 is thrown and loses otherwise. Finally, the **craps 3 bet** pays 15 : 1 if a 3 is thrown and loses otherwise

15. Let  $W$  denote the winnings for a unit craps bet. Show that

- $\mathbb{P}(W = -1) = \frac{8}{9}, \mathbb{P}(W = 7) = \frac{1}{9},$
- $\mathbb{E}(W) \approx -0.1111$
- $\text{sd}(W) \approx 2.5142$

16. In the **craps experiment**, select the craps bet. Run the simulation 1000 times, updating every 10 runs and watch the apparent convergence of the empirical density and moments of  $W$  to the true density and moments. Suppose that you

bet \$1 on each of the 1000 games. What would your net winnings be?

17. Let  $W$  denote the winnings for a unit craps 2 bet or a unit craps 12 bet. Show that

- $\mathbb{P}(W = -1) = \frac{35}{36}, \mathbb{P}(W = 30) = \frac{1}{36},$
- $\mathbb{E}(W) \approx -0.1389$
- $\text{sd}(W) \approx 5.0944$

18. In the **craps experiment**, select the craps 2 bet. Run the simulation 1000 times, updating every 10 runs and watch the apparent convergence of the empirical density and moments of  $W$  to the true density and moments. Suppose that you bet \$1 on each of the 1000 games. What would your net winnings be?

19. In the **craps experiment**, select the craps 12 bet. Run the simulation 1000 times, updating every 10 runs and watch the apparent convergence of the empirical density and moments of  $W$  to the true density and moments. Suppose that you bet \$1 on each of the 1000 games. What would your net winnings be?

20. Let  $W$  denote the winnings for a unit craps 3 bet. Show that

- $\mathbb{P}(W = -1) = \frac{17}{18}, \mathbb{P}(W = 15) = \frac{1}{18},$
- $\mathbb{E}(W) \approx -0.1111$
- $\text{sd}(W) \approx 3.6650$

21. In the **craps experiment**, select the craps 3 bet. Run the simulation 1000 times, updating every 10 runs and watch the apparent convergence of the empirical density and moments of  $W$  to the true density and moments. Suppose that you bet \$1 on each of the 1000 games. What would your net winnings be?

### Big Six and Big Eight

The **big 6 bet** is a bet that 6 is thrown before 7. Similarly, the **big 8 bet** is a bet that 8 is thrown before 7. Both pay even money 1 : 1.

22. Let  $W$  denote the winnings for a unit big 6 bet or a unit big 8 bet. Show that

- $\mathbb{P}(W = -1) = \frac{6}{11}, \mathbb{P}(W = 1) = \frac{5}{11},$
- $\mathbb{E}(W) \approx -0.0909$
- $\text{sd}(W) \approx 0.9959$

23. In the **craps experiment**, select the big 6 bet. Run the simulation 1000 times, updating every 10 runs and watch the apparent convergence of the empirical density and moments of  $W$  to the true density and moments. Suppose that you bet \$1 on each of the 1000 games. What would your net winnings be?

24. In the **craps experiment**, select the big 8 bet. Run the simulation 1000 times, updating every 10 runs and watch the apparent convergence of the empirical density and moments of  $W$  to the true density and moments. Suppose that you bet \$1 on each of the 1000 games. What would your net winnings be?

## Hardway Bets

A **hardway bet** can be made on any of the numbers 4, 6, 8, or 10. It is a bet that the chosen number  $n$  will be thrown “the hardway” as  $\left(\frac{n}{2}, \frac{n}{2}\right)$ , before 7 is thrown and before the chosen number is thrown in any other combination. Hardway bets on 4 and 10 pay 7 : 1, while hardway bets on 6 and 8 pay 9 : 1.

25. Let  $W$  denote the winnings for a unit hardway 4 or hardway 10 bet. Show that

- $\mathbb{P}(W = -1) = \frac{8}{9}, \mathbb{P}(W = 7) = \frac{1}{9},$
- $\mathbb{E}(W) \approx -0.1111$
- $\text{sd}(W) \approx 2.5142$

26. In the **craps experiment**, select the hardway 4 bet. Run the simulation 1000 times, updating every 10 runs and watch the apparent convergence of the empirical density and moments of  $W$  to the true density and moments. Suppose that you bet \$1 on each of the 1000 games. What would your net winnings be?

27. In the **craps experiment**, select the hardway 10 bet. Run the simulation 1000 times, updating every 10 runs and watch the apparent convergence of the empirical density and moments of  $W$  to the true density and moments. Suppose that you bet \$1 on each of the 1000 games. What would your net winnings be?

28. Let  $W$  denote the winnings for a unit hardway 6 or hardway 8 bet. Show that

- $\mathbb{P}(W = -1) = \frac{10}{11}, \mathbb{P}(W = 9) = \frac{1}{11},$
- $\mathbb{E}(W) \approx -0.0909$
- $\text{sd}(W) \approx 2.8748$

29. In the **craps experiment**, select the hardway 6 bet. Run the simulation 1000 times, updating every 10 runs and watch the apparent convergence of the empirical density and moments of  $W$  to the true density and moments. Suppose that you bet \$1 on each of the 1000 games. What would your net winnings be?

30. In the **craps experiment**, select the hardway 8 bet. Run the simulation 1000 times, updating every 10 runs and watch the apparent convergence of the empirical density and moments of  $W$  to the true density and moments. Suppose that you bet \$1 on each of the 1000 games. What would your net winnings be?

## The Distribution of the Number of Rolls

Next let us compute the distribution and moments of the number of rolls  $N$  in a game of craps. This random variable is of no special interest to the casino or the players, but provides a good mathematical exercise. By definition, if the shooter wins or loses on the first roll,  $N = 1$ . Otherwise, the shooter continues until she either makes her point or rolls 7. In this latter case, we can use the **geometric distribution** on  $\mathbb{N}_+$  which governs the trial number of the first success in a sequence of **Bernoulli trials**.

31. Show that  $\mathbb{P}(N = 1 | Z_1 = z) = 1$  if  $z \in \{2, 3, 7, 11, 12\}$

32. Show that  $\mathbb{P}(N = n|Z_1 = z) = p_z (1 - p_z)^{n-2}$ ,  $n \in \{2, 3, \dots\}$  for the values of  $z$  and  $p_z$  given in the table below. Thus, the conditional distribution of  $N - 1$  given  $Z = z$  is geometric with parameter  $p_z$ .

$z$	4	5	6	8	9	10
$p_z$	$\frac{9}{36}$	$\frac{10}{36}$	$\frac{11}{36}$	$\frac{11}{36}$	$\frac{10}{36}$	$\frac{9}{36}$

The distribution of  $N$  is a [mixture of distributions](#).

33. Use the result of the previous exercises to show that

$$\mathbb{P}(N = n) = \begin{cases} \frac{12}{36}, & n = 1 \\ \frac{1}{24} \left(\frac{3}{4}\right)^{n-2} + \frac{5}{81} \left(\frac{13}{18}\right)^{n-2} + \frac{55}{648} \left(\frac{25}{36}\right)^{n-2}, & n \in \{2, 3, \dots\} \end{cases}$$

34. Simplify numerically to find that the first few values of the probability density function of  $N$ :

$n$	1	2	3	4	5
$\mathbb{P}(N = n)$	0.33333	0.18827	0.13477	0.09657	0.06926

35. Find the probability that a game of craps will last at least 8 rolls.



36. Use conditional probability arguments and the moments of the geometric distribution to show that

- $\mathbb{E}(N) = 3.3758$
- $\mathbb{E}(N^2) = 15.0013$
- $\text{var}(N) = 3.6056$
- $\text{sd}(N) = 1.8988$