

# Kalkulus tizenhetedik feladatsor - megoldás

Parciális integrálás, racionális törtfüggvények integrálása

1. Számítsa ki az alábbi határozatlan integrálokat!

- a)  $\int x \cos(2x) dx$
- b)  $\int (x^2 + 1)e^x dx$
- c)  $\int (3x - 1) \sin(5x + 3) dx$  (K87)
- d)  $\int x^3 \ln(2x) dx$  (K87)
- e)  $\int \ln(x) dx$
- f)  $\int \arctan(2x) dx$  (K87)
- g)  $\int e^x \sin(x) dx$
- h)  $\int \cosh(2x) \sin(5x) dx$  (K88)
- i)  $\int \ln(5x) dx$
- j)  $\int (2x + 5) \ln(5x) dx$
- k)  $\int (5x + 2) \sinh(4x) dx$
- l)  $\int x^2 \cos(3x) dx$
- m)  $\int \arcsin(2x) dx$
- n)  $\int 4x \arctan(2x) dx$

*Megoldás:* A parciális integrálás képlete:  $\int f'g = fg - \int fg'$

$$i) \int \ln(5x) dx = \int \overbrace{1}^{f'} \cdot \overbrace{\ln(5x)}^g dx = x \ln(5x) - \int x \frac{1}{5x} 5 dx = x \ln(5x) - \int 1 dx = x \ln(5x) - x + C$$

$$j) \int (2x + 5) \ln(5x) dx = \int \overbrace{(2x + 5)}^{f'} \cdot \overbrace{\ln(5x)}^g dx, \text{ Ekkor } f(x) = x^2 + 5x \\ \int (2x + 5) \ln(5x) dx = (x^2 + 5x) \ln(5x) - \int (x^2 + 5x) \frac{1}{5x} 5 dx = \\ (x^2 + 5x) \ln(5x) - \int x + 5 dx = (x^2 + 5x) \ln(5x) - \left(\frac{x^2}{2} + 5x\right) + C$$

$$k) \int (5x + 2) \sinh(4x) dx = \int \overbrace{(5x + 2)}^g \cdot \overbrace{\sinh(4x)}^{f'} dx \text{ Ekkor } f = \frac{\cosh(4x)}{4} \\ \int (5x + 2) \sinh(4x) dx = (5x + 2) \frac{\cosh(4x)}{4} - \int \frac{\cosh(4x)}{4} 5 dx = (5x + 2) \frac{\cosh(4x)}{4} - \frac{\sinh(4x)}{16} 5 + C$$

$$l) \int x^2 \cos(3x) dx = \int \overbrace{x^2}^g \cdot \overbrace{\cos(3x)}^{f'} dx \text{ Ekkor } f = \frac{\sin(3x)}{3}, \\ \int x^2 \cos(3x) dx = x^2 \frac{\sin(3x)}{3} - \int 2x \frac{\sin(3x)}{3} dx$$

$$\int 2x \frac{\sin(3x)}{3} dx = \frac{2}{3} \int \overbrace{x}^g \cdot \overbrace{\sin(3x)}^{f'} dx \text{ Ekkor } f = -\frac{\cos(3x)}{3}, \\ \int 2x \frac{\sin(3x)}{3} dx = \frac{2}{3} \left( -\frac{x \cos(3x)}{3} - \int -\frac{\cos(3x)}{3} dx \right) = \\ = -\frac{2x \cos(3x)}{9} + \frac{2 \sin(3x)}{27} + C \\ \int x^2 \cos(3x) dx = x^2 \frac{\sin(3x)}{3} + \frac{2x \cos(3x)}{9} - \frac{2 \sin(3x)}{27} + C$$

$$\begin{aligned}
m) \int \arcsin(2x) dx &= \int \overbrace{1}^{f'} \cdot \overbrace{\arcsin(2x)}^g dx = \\
&= x \arcsin(2x) - \int x \frac{1}{\sqrt{1-(2x)^2}} 2 dx = x \arcsin(2x) - \left(-\frac{1}{4}\right) \int \underbrace{(-8x)}_{f'} \underbrace{(1-4x^2)^{-\frac{1}{2}}}_{f^\alpha} dx = \\
&= x \arcsin(2x) + \frac{1}{4} \frac{(1-4x^2)^{\frac{1}{2}}}{\frac{1}{2}} + C = x \arcsin(2x) + \frac{1}{2} (1-4x^2)^{\frac{1}{2}} + C
\end{aligned}$$

$$\begin{aligned}
n) \int 4x \arctan(2x) dx &= \int \overbrace{4x}^{f'} \cdot \overbrace{\arctan(2x)}^g dx \text{ Ekkor } f = 2x^2 \\
\int 4x \arctan(2x) dx &= 2x^2 \arctan(2x) - \int 2x^2 \frac{1}{1+(2x)^2} 2 dx = 2x^2 \arctan(2x) - \\
&\int \frac{4x^2+1-1}{1+4x^2} dx = 2x^2 \arctan(2x) - \int 1 - \frac{1}{1+4x^2} dx = \\
&= 2x^2 \arctan(2x) - x + \frac{\arctan(2x)}{2} + C
\end{aligned}$$

2. Számítsa ki az alábbi racionális törtfüggvények integrálját! (Kónya 5.3 fejezet.)

- a)  $\int \frac{1}{x^2+2x-3} dx$
- b)  $\int \frac{x+1}{x^2+3x} dx$  (Mo: Kónya 89.o.)
- c)  $\int \frac{2x+1}{x^2-5x+6} dx$  (Mo: Kónya 90.o.)
- d)  $\int \frac{1}{x^3+2x^2} dx$  (Mo: Kónya 90.o.)
- e)  $\int \frac{x+1}{(x-1)^2(x-3)} dx$  (Mo: Kónya 91.o.)
- f)  $\int \frac{x^3}{x^4-16} dx$  (Mo: Kónya 91.o.)
- g)  $\int \frac{x^5-15x}{x^4-16} dx$  (Mo: Kónya 91.o.)

*Megoldás:* A polinomosztáshoz:  $(x^5 - 15x) : (x^4 - 16)$

$$\begin{array}{r}
(x^5 - 15x) \div (x^4 - 16) = x + \frac{x}{x^4 - 16} \\
-x^5 + 16x \\
\hline
x
\end{array}$$

*Másik példa:*  $(x^3 - 1) : (x - 1)$

$$\begin{array}{r}
(x^3 - 1) \div (x - 1) = x^2 + x + 1 \\
-x^3 + x^2 \\
\hline
x^2 \\
-x^2 + x \\
\hline
x - 1 \\
-x + 1 \\
\hline
0
\end{array}$$

3. Számítsa ki az alábbi határozatlan integrálokat!

a)  $\int \frac{x^2}{x^2-9} dx$

b)  $\int \frac{1}{(x-1)^2} dx$

c)  $\int \frac{1}{x^2-1} dx$

d)  $\int \frac{x^2}{x^3-1} dx$

e)  $\int \frac{1}{x^3-1} dx$

*Megoldás:*

a)  $\int \frac{x^2}{x^2-9} dx = \int \frac{x^2-9+9}{x^2-9} dx = \int 1-9\frac{1}{(x-3)(x+3)} dx = x-9 \int \frac{1}{(x-3)(x+3)} dx$   
*Ekkor*

$$\frac{1}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$$

$$1 = A(x+3) + B(x-3)$$

Ha  $x = 3$ ,  $1 = A6$ ,  $\Rightarrow A = \frac{1}{6}$ , valamint ha  $x = -3$ ,  $1 = B(-6)$ ,  
 $\Rightarrow B = -\frac{1}{6}$ . *Ebből:*

$$\begin{aligned} \int \frac{1}{(x-3)(x+3)} dx &= \int \frac{\frac{1}{6}}{(x-3)} - \frac{\frac{1}{6}}{(x+3)} dx = \\ &= \frac{1}{6} \int \frac{1}{(x-3)} - \frac{1}{(x+3)} dx = \frac{1}{6} (\ln|x-3| - \ln|x+3|) + C \end{aligned}$$

*Így:*

$$\int \frac{x^2}{x^2-9} dx = x - \frac{9}{6} \ln|x-3| + \frac{9}{6} \ln|x+3| + C = x - \ln \left| \frac{x+3}{x-3} \right|^{\frac{3}{2}} + C$$

b)  $\int \frac{1}{(x-1)^2} dx = \int (x-1)^{-2} dx = \frac{(x-1)^{-1}}{-1} + C = \frac{1}{1-x} + C$

c)  $\int \frac{1}{x^2-1} dx = \int \frac{1}{(x-1)(x+1)} dx$  *Ekkor*

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$1 = A(x+1) + B(x-1)$$

Ha  $x = 1$ ,  $1 = A2$ ,  $\Rightarrow A = \frac{1}{2}$ , valamint ha  $x = -1$ ,  $1 = B(-2)$ ,  
 $\Rightarrow B = -\frac{1}{2}$ . *Ebből:*

$$\begin{aligned} \int \frac{1}{(x-1)(x+1)} dx &= \int \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1} dx = \frac{1}{2} (\ln|x-1| - \ln|x+1|) + C = \\ &= \ln \sqrt{\left| \frac{x-1}{x+1} \right|} + C \end{aligned}$$

$$\int \frac{f'}{f} = \ln|f|$$

d)  $\int \frac{x^2}{x^3-1} dx = \int \frac{1}{3} \frac{3x^2}{x^3-1} dx = \frac{1}{3} \ln|x^3-1| + C =$   
 $= \ln \sqrt[3]{|x^3-1|} + C$

e)  $\int \frac{1}{x^3-1} dx = \int \frac{1}{(x-1)(x^2+x+1)} dx$  *Ekkor*

$$\frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$1 = A(x^2+x+1) + Bx(x-1) + C(x-1)$$

Ha  $x = 1$ ,  $1 = A3$ ,  $\Rightarrow A = \frac{1}{3}$ , ha  $x = 0$ ,  
 $1 = A + C(-1) = \frac{1}{3} - C$ ,  $\Rightarrow C = -\frac{2}{3}$ . *Valamint:*

$$\begin{aligned} 1 &= \frac{1}{3}(x^2 + x + 1) + B(x^2 - x) - \frac{2}{3}(x - 1) \\ 1 &= \frac{1}{3}x^2 + \frac{1}{3}x + \frac{1}{3} + Bx^2 - Bx - \frac{2}{3}x + \frac{2}{3} \\ 1 &= \left(\frac{1}{3} + B\right)x^2 + \left(\frac{1}{3} - B - \frac{2}{3}\right)x + 1 \end{aligned}$$

Ahonnán  $B = -\frac{1}{3}$

$$\begin{aligned} \int \frac{1}{(x-1)(x^2+x+1)} dx &= \int \frac{\frac{1}{3}}{x-1} + \frac{-\frac{1}{3}x - \frac{2}{3}}{x^2+x+1} dx = \\ &= \frac{1}{3} \int \frac{1}{x-1} - \frac{x+2}{x^2+x+1} dx = \frac{1}{3} \ln|x-1| - \frac{1}{3} \int \frac{1}{2x^2+x+1} + \frac{3}{2x^2+x+1} dx = \\ &= \frac{1}{3} \ln|x-1| - \frac{1}{6} \ln|x^2+x+1| - \frac{1}{2} \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} dx \end{aligned}$$

Ahol az utolsó tag:

$$\begin{aligned} \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} dx &= \int \frac{4}{3} \frac{1}{\frac{\left(x+\frac{1}{2}\right)^2}{\frac{3}{4}} + 1} dx = \frac{4}{3} \int \frac{1}{\left(\frac{2x+1}{\sqrt{3}}\right)^2 + 1} dx = \\ &= \frac{4}{3} \frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{\frac{2}{\sqrt{3}}} + C = \frac{2}{\sqrt{3}} \arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C \end{aligned}$$

Így:

$$\int \frac{1}{x^3-1} dx = \frac{1}{3} \ln|x-1| - \frac{1}{6} \ln|x^2+x+1| - \frac{1}{\sqrt{3}} \arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$