Kalkulus vizsga 3 Mo — 2021. jan

1.a) (10p) Rendőr-elv (2)

$$\frac{1}{\sqrt[n]{9}\cdot\sqrt[n]{n}} = \frac{\sqrt[n]{1}}{\sqrt[n]{9n}} = \sqrt[n]{\frac{1}{9n}} \underset{(2)}{\le} \sqrt[n]{\frac{1}{6n+3}} \underset{(2)}{\le} \sqrt[n]{\frac{1}{1}} = \sqrt[n]{1} = 1$$

$\sqrt[n]{9} \to 1$, $\sqrt[n]{n} \to 1$ (2) $\Rightarrow \frac{1}{\sqrt[n]{9}\cdot\sqrt[n]{n}} \to 1$

$6n+3n \ge 6n+3$, $n \in \mathbb{N}^+$

$6n+3 \ge 1$

$1 \to 1$ (1)

$$\Longrightarrow \sqrt[n]{\frac{1}{6n+3}} \xrightarrow{(1)} 1 \quad (n \to +\infty)$$

1.b) (10p)

$$\lim_{n\to+\infty} \frac{3^n + 100n^7 + 324}{3^{n+1} - \sqrt{n} + \log(n)} = \lim_{n\to+\infty} \frac{\overset{\text{dom (1)}}{3^n} + 100n^7 + 324}{\underset{\text{dom (1)}}{3^n}\cdot 3 - \sqrt{n} + \log(n)} =$$

Erősorrend: $\log(n) \ll \sqrt{n} \ll n^7 \ll 3^n$ (3)

$$= \lim_{n\to+\infty} \underbrace{\frac{3^n}{3^n}}_{=1\ (1)} \cdot \frac{1 - 100\cdot\frac{n^7}{3^n} + 324\cdot\frac{1}{3^n}}{3 - \frac{\sqrt{n}}{3^n} + \frac{\log(n)}{3^n}} = 1\cdot\frac{1}{3} = \frac{1}{3} \ (1)$$

$\frac{n^7}{3^n} \to 0$, $\frac{1}{3^n} \to 0$, $\frac{\sqrt{n}}{3^n} \to 0$, $\frac{\log(n)}{3^n} \to 0$

alak: (2) $\quad \forall$ tart 0: (1)

2.) (10p)

$(A-B)(A+B) = A^2 - B^2$ ötlet: (2)

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{1-\cos x}{x^2}\cdot\frac{1+\cos x}{1+\cos x} =$$

"$\frac{0}{0}$" (1)

$$= \lim_{x\to 0} \frac{\overset{(1)}{1-\cos^2 x}}{x^2\cdot(1+\cos x)} = \lim_{x\to 0} \overbrace{\frac{\sin^2 x}{x^2}\cdot\frac{1}{1+\cos x}}^{(1)} =$$

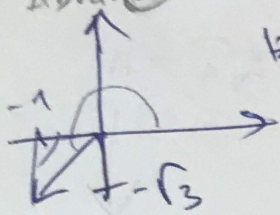
$\sin^2 x + \cos^2 x = 1$ (1)

$$= \lim_{x\to 0} \underbrace{\left(\frac{\sin x}{x}\right)^2}_{\to 1\ (1)} \cdot \frac{1}{1+\cos x} = 1^2\cdot\overbrace{\frac{1}{1+\cos 0}}^{(1)} = 1\cdot\frac{1}{2} = \frac{1}{2} \ (1)$$

VAGY: L'Hospitallal is jó!

3.) (15p) $z = -1-\sqrt{3}i$ kell z trigonometrikus alakja

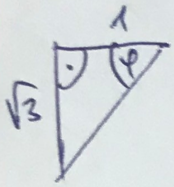
Ábra: 3



$|z| = r = \sqrt{(-1)^2+(-\sqrt{3})^2} = \sqrt{4} = 2$ 2

z trigo. alakja:

$z = -1-\sqrt{3}i = 2\cdot\left(\cos\frac{4\pi}{3} + i\cdot\sin\frac{4\pi}{3}\right)$



$tg\,\varphi = \frac{\sqrt{3}}{1} \Rightarrow \varphi = \frac{\pi}{3} = 60°$

z szöge az ábra alapján: $\alpha = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$ 3

$z^3 = r^3\cdot(\cos(3\alpha) + i\cdot\sin(3\alpha)) = \underbrace{2^3\cdot(\cos 4\pi + i\sin 4\pi)}_{3}$ 2

$= 8\cdot(\underbrace{\cos 0}_{1} + i\cdot\underbrace{\sin 0}_{0}) = \underline{\underline{8}}$ 2

2π szerint per.

4.) (20p) $f(x) = 1-2\cdot\ln(x^2+2x+2)$ $\quad I = [-2,2]$

$D_f = \mathbb{R}$, mert kell: $x^2+2x+2 = (x+1)^2+1 > 0 \quad \forall x\in\mathbb{R}$. 1 2

4 f-nek ott lehet szélsőértéke I-n, ahol $f'(x)=0$ vagy az int. végpontjaiban. Weierstrass-tétel miatt ∃ min, max I-n (f folyt.) 2

$f'(x) = -2\cdot\frac{1}{x^2+2x+2}\cdot 2x+2 = \underbrace{\frac{-4x-4}{x^2+2x+2}}_{2} = 0 \Leftrightarrow$

$-4x = 4$ 1

$x = -1 \in I$ 1

<u>Vesszük:</u>

$f(-2) = 1-2\cdot\ln(4-4+2) = 1-2\cdot\overbrace{\ln(2)}^{>0}$ 1

$f(-1) = 1-2\cdot\ln(1-2+2) = 1-2\cdot\overbrace{\ln(1)}^{0} = 1$ 1 $\boxed{MAX}$ 1

$f(2) = 1-2\cdot\ln(4+4+2) = 1-2\cdot\underbrace{\ln(10)}_{>0}$ 1 $\boxed{MIN}$ 1

Az ln fv. szig. mon. nő: $\ln(2) < \ln(10)$ és $0 < \ln(2) < \ln(10)$ 2

5.) (10p) Az érintő egyenlete x_0-ban:

$$y = f'(x_0)\cdot(x-x_0)+f(x_0) \quad ① \qquad x_0=\pi$$

$$f(x_0)=f(\pi)=\cos(-\pi)+\pi=-1+\pi \quad ②$$

$$f(x)=\cos(-x)+x$$

$$f'(x)=-\sin(-x)\cdot(-1)+1=\sin(-x)+1 \quad ②$$

$$f'(x_0)=f'(\pi)=\sin(-\pi)+1=1 \quad ②$$

Tehát az érintő egyenlete:

$$y=1\cdot(x-\pi)-1+\pi \overset{②}{=} x-\pi-1+\pi=x-1$$

$$e:\ y=x-1. \quad ①$$

6.) a) (15p)

$$\int \underbrace{x}_{f(x)}\cdot\underbrace{\sin(2x+1)}_{g'(x)}\,dx = \overbrace{\underbrace{x}_{f}\cdot\underbrace{\frac{-\cos(2x+1)}{2}}_{g}-\int \underbrace{1}_{f'}\cdot\underbrace{\frac{-\cos(2x+1)}{2}}_{g}}^{③} =$$

parc. int. ①

$f'(x)=1$ ③ $\quad g(x)=\int g'(x)=\int \sin(2x+1)=\dfrac{-\cos(2x+1)}{2}$ ②

$\int f\cdot g'=f\cdot g-\int f'\cdot g$ ①

$\int h(ax+b)=\dfrac{1}{a}H(ax+b)$ ① ahol $H'=h$.

$$=-x\cdot\frac{\cos(2x+1)}{2}+\frac{1}{2}\underbrace{\int\cos(2x+1)\,dx}_{\frac{\sin(2x+1)}{2}\ ②} =$$

$$\underbrace{=-x\cdot\frac{\cos(2x+1)}{2}+\frac{1}{4}\cdot\sin(2x+1)+C,\quad C\in\mathbb{R}\ \text{tetsz.}}_{②}$$

6. b) (10p)

$$\int x^2 \cdot \sqrt{4-x^3} = \int x^2 \cdot (4-x^3)^{1/2}\,dx = \text{①}$$

$$\overset{①}{=} -\frac{1}{3}\int -3x^2 \cdot (4-x^3)^{1/2}\,dx = \underbrace{-\frac{1}{3}\cdot\frac{(4-x^3)^{3/2}}{3/2}}_{②} + C \; (=)$$

$$\int f' \cdot f^{\alpha} = \frac{f^{\alpha+1}}{\alpha+1} \quad ②$$

$$\left.\begin{array}{l} \alpha = 1/2 \\ f(x) = 4-x^3, \; f'(x) = -3x^2 \end{array}\right\} ③$$

$$(=) -\frac{1}{3}\cdot\frac{2}{3}\cdot(4-x^3)^{3/2} + C = \underbrace{-\frac{2}{9}(4-x^3)^{3/2} + C}_{①}$$

CÉR tetra.

+Fe (15p)

$$\int_2^{+\infty} \frac{1}{(x-1)(x+6)}\,dx \quad \text{impr. int.}$$

parc. törtekre bontás: ①

$$\frac{1}{(x-1)(x+6)} \overset{①}{=} \frac{A}{x-1} + \frac{B}{x+6} = \frac{A(x+6)+B(x-1)}{(x-1)(x+6)} =$$

$$= \frac{Ax+6A+Bx-B}{(x-1)(x+6)} = \frac{\boxed{(A+B)x+6A-B}}{(x-1)(x+6)} \quad ②$$

$$\Leftrightarrow 1 = (A+B)x + 6A - B \Leftrightarrow \begin{array}{l} 0 = A+B \Rightarrow A = -B \\ 1 = 6A - B \quad \Downarrow \; 1 = -6B - B = -7B \end{array}$$

$$\underbrace{B = -\frac{1}{7} \quad A = \frac{1}{7}}_{②} \; k \Downarrow$$

Tehát

$$\int_{+2}^{+\infty} \frac{1}{(x-1)(x+6)}\,dx \overset{②}{=} \lim_{k\to+\infty} \int_2^k \frac{1}{(x-1)(x+6)} = \lim_{k\to+\infty} \frac{1}{7}\int_2^k \frac{1}{x-1} - \frac{1}{x+6}\,dx \; ①$$

$$= \lim_{k\to+\infty} \frac{1}{7}\left[\ln|x-1| - \ln|x+6|\right]_{x=2}^{k} \overset{②}{=} \lim_{k\to+\infty} \frac{1}{7}\left[\ln\left|\frac{x-1}{x+6}\right|\right]_{x=2}^{k} \; ①$$

$$= \lim_{k\to+\infty} \frac{1}{7}\cdot\left(\ln\left|\underbrace{\frac{k-1}{k+6}}_{\to 1}\right| - \ln\left|\frac{2-1}{2+6}\right|\right) =$$

1

$$= \frac{1}{7}\cdot\underbrace{\left(\underbrace{\ln 1}_{0} - \ln\left|\frac{1}{8}\right|\right)}_{1} = -\frac{1}{7}\cdot \underbrace{\ln\frac{1}{8}}_{\underbrace{\ln 1}_{0} - \ln 8} = \underline{\underline{\frac{1}{7}\cdot\ln 8}}$$

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