THESIS

Mandelbrot percolations with inhomogeneous probabilities

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Chapter 1

Preliminaries

1.1 Introduction

Fractals and self-similar sets are the topic of common interest for long time. In the twentieth century mathematicians – mainly Benoit Mandelbrot noticed that most of the fractals in nature or in real life are random fractals. This thesis focuses on the so-called Mandelbrot percolation fractals or random percolation sets, a family of random fractals. In a nutshell we have an initial set, and we retain or throw away certain subsets of this initial set with given probabilities and in the next level we do the same thing with the retained squares and so on, for a precise definition see chapter 1.2.1. In our case the initial set, and also the subsets are squares, and mostly the probabilities are homogeneous meaning that they are all the same. In this thesis we give a survey about some of the geometric measure theoretic properties in the above mentioned homoegeneous case and some result from the last few years in the inhomogeneous case. It is worth mentioning that Shmerkin and Suomala [10]

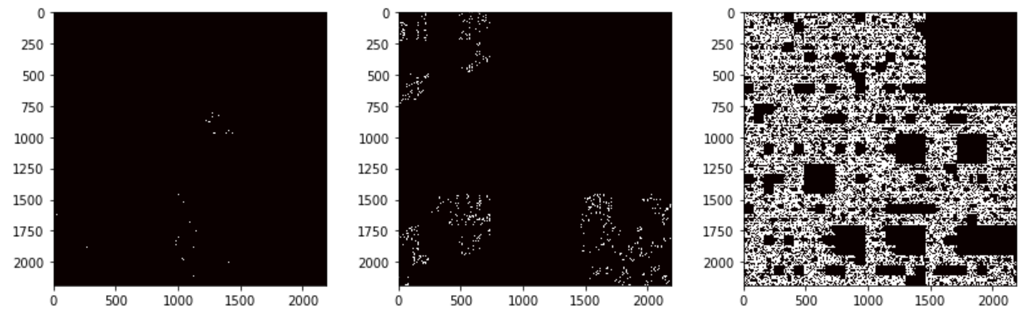

Figure 1.1: Seventh level approximation of realizations with $M = 3$ and probabilities p_1=((1/3,1/3,1/3),(1/3,1/3,1/3),(1/3,1/3,1/3)); p_2=((0.5,0.4,0.7),(0.5,0.4,0.9),(0.6,0.5,0.5)); p_3=((1,0.9,0.89),(0.7,0.9,0.6),(1,0.9,0.98)).

gave a very detailed description on the properties of the homogeneous case, but this text is rather focus on the antecedents of this big result.

1.2 Mandelbrot percolation

1.2.1 Construction of the Mandelbrot percolation fractal

Let $I := [0,1]^2$ denote the unit square. For given $M \geq 2$ integer and $p_{i,j} \in [0,1] \quad \forall (i,j) \in \{0,1,...,M-1\}^2$ probabilities the Mandelbrot percolation set in the 2-dimensional Euclidean-space is constructed in the following way: let $\mathcal{T}_n := \{(\underline{i}_n, \underline{j}_n) \quad | \quad \underline{i}_n, \underline{j}_n \in \{0,1,...,M-1\}^n\}$ denote the pairs of n-length sequences from $\{0,1,...,M-1\}$ indexing the level n sub-squares of I, the empty sequence is denoted by $\emptyset$, as follows $\mathcal{T}_0 = (\emptyset, \emptyset)$. Denote the first level

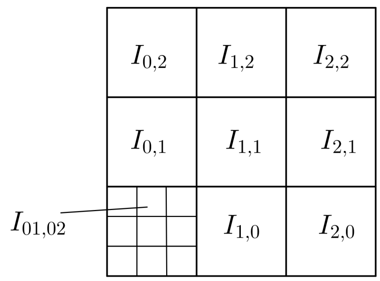

Figure 1.2: The partition.

sub-squares of I of side length $\frac{1}{M}$ with $I_{i,j}$:

$$I_{i,j} := \left[\frac{i}{M}, \frac{i+1}{M}\right] \times \left[\frac{j}{M}, \frac{j+1}{M}\right]$$

This is a partition of the unit square into M^2 congruent squares :

$$I = \bigcup_{i,j=0}^{M-1} I_{i,j}.$$

We can define the level n squares similarly: if $(\underline{i}_n, \underline{j}_n) \in \mathcal{T}_n$ then

$$I_{\underline{i}_n,\underline{j}_n} = \left[\sum_{k=1}^{n} i_k \cdot \frac{1}{M^k}, \sum_{k=1}^{n} i_k \cdot \frac{1}{M^k} + \frac{1}{M^n}\right] \times \left[\sum_{k=1}^{n} j_k \cdot \frac{1}{M^k}, \sum_{k=1}^{n} j_k \cdot \frac{1}{M^k} + \frac{1}{M^n}\right].$$

Now we have the base for the fractal percolation set. The next step is to define the survival set $\mathcal{E}_n$ consists of the index of the retained level n squares.

Definition 1.1. *$\mathcal{E}_0 = \mathcal{T}_0 = (\emptyset, \emptyset)$ and inductively if we have $\mathcal{E}_{n-1}$ and $(\underline{i}_{n-1}, \underline{j}_{n-1}) \notin \mathcal{E}_{n-1}$ then for all $(i,j) \in \{0,1,...,M-1\}^2$ $((i_1,...,i_{n-1},i),$ $(j_1,...,j_{n-1},j)) \notin \mathcal{E}_n$, if $(\underline{i}_{n-1}, \underline{j}_{n-1}) \in \mathcal{E}_{n-1}$ then $((i_1,..,i_{n-1},i),(j_1,..,j_{n-1},j))$ $\in \mathcal{E}_n$ with probability $p_{i,j}$.*

We can also think about $\mathcal{T}_n$ as an M^2-ary tree with height n and nodes $(\underline{i}_k, \underline{j}_k)$. An $(\underline{i}_k, \underline{j}_k)$ node has M^2 children: $(\underline{i}_k i, \underline{j}_k j)$ $i, j \in M$. For $p = (p_{0,0}, ..., p_{M-1,M-1})$ we can introduce a probability measure $\mathbb{P}_p$ on the space of labeled trees. For each node $(i_1...i_n, j_1...j_n)$ we give a random label $X_{i_1...i_n, j_1...j_n}$ this will be 0 or 1. It is required that

1. $X_{i_1...i_n, j_1...j_n}$ are independent Bernoulli random variables;
2. $\mathbb{P}(X_\emptyset) = 1$;
3. $\mathbb{P}_p(X_{i_1...i_n, j_1..j_n}) = p_{i_n, j_n}$.

Thus $\mathcal{E}_n = \{i_1...i_n, j_1...j_n : X_{i_1,j_1} = X_{i_1 i_2, j_1 j_2} = ... = X_{i_1...i_n, j_1...j_n} = 1\}$. Now the n^{th} level approximation of E is E_n, defined by the survival set $\mathcal{E}_n$:

$$E_n = \bigcup_{(\underline{i}_n, \underline{j}_n) \in \mathcal{E}_n} I_{\underline{i}_n, \underline{j}_n} \text{ and from that } \quad E = \bigcap_{n=1}^{\infty} E_n.$$

The above defined E is random variable i.e. E: $\Omega \to \{\text{the Cantor sets of } I^2\}$, where Ω is an infinite randomly labeled tree, defined above.

Homogeneous and inhomogeneous case

I distinguish two cases, the homogeneous and the inhomogeneous, the first is when all the squares are chosen with the same probability, so $\forall (i,j)$ $p_{i,j} = p$, and the second is when the probabilities are not the same.

Independence

Although at every level the squares are selected or discarded independently of each other it is not true that at a certain level the event that two distinct square is retained is independent – they are only conditionally independent. For example look at the case when $M = 3$ and the probabilities are the same p. $\mathbb{P}(I_{11,11} \subset E_2, I_{11,12} \subset E_2) = \mathbb{P}$(selecting the square with index 1,1 and than selecting the small square with index 11, 11 and 11,12)=p^3 but $\mathbb{P}(I_{11,11} \subset E_2) = \mathbb{P}(I_{11,12} \subset E_2) = p^2$.

1.3 A brief introduction to fractal geometry

In this section I introduce two essential concept of fractal geometry namely the Hausdorff measure and dimension, and the Box dimension.

Hausdorff measure and dimension

Let U be any non-empty subset of the Euclidean space, $\mathbb{R}^n$, $diam(U) = \sup\{|x-y| : x, y \in U\}$. We call $\{U_i\}_{i\in I}$ a countable collection of sets a δ -cover of U if $\forall i \in I$ $diam(U_i) < \delta$ and $U \subset \bigcup\limits_{i\in I} U_i$. For $\delta > 0$ we define $\mathscr{H}^s_\delta(U) = \inf\{\sum_{i\in I} diam(U_i)^s : \{U_i\}_{i\in I}$is a δ-cover of $U\}$.

Definition 1.2. *The s-dimensional Hausdorff measure of U is*

$$\mathscr{H}^s(U) = \lim_{\delta\to 0} \mathscr{H}^s_\delta(U). \tag{1.1}$$

The limit exists, because as δ decreases the infimum increases and approaches a limit, which limit is usually infinity or zero. If we take a look at the graph of $\mathscr{H}^s(U)$ than we'll see that there is a critical value of s at which $\mathscr{H}^s(U)$ jumps from infinity to zero. This critical value is called the Hausdorff dimension of the set U. For a more precise explanation see [3].

Definition 1.3. *The Hausdorff dimension of a set U is*

$$\dim_H U = \inf\{s : \mathscr{H}^s(U) = 0\} = \sup\{s : \mathscr{H}^s(U) = \infty\}. \tag{1.2}$$

The following property of Hausdorff dimension and measure will be used in Chapter 2.

Proposition 1.1. *Let $U \subset \mathbb{R}^n$, $f : U \to \mathbb{R}^m$ be a mapping with Hölder condition of exponent α i.e. $|f(x) - f(y)| \leq c \cdot |x - y|^\alpha$. Then $\forall s \in \mathbb{R}$ $\mathscr{H}^{s/\alpha}(f(U)) \leq c^{s/\alpha}\mathscr{H}^s(U)$.*

For the proof see [3]. The next proposition easily follows from that:

Proposition 1.2. *Let $U \subset \mathbb{R}^n$, $f : U \to \mathbb{R}^m$ be a mapping with Hölder condition of exponent α. Then $\dim_H f(U) \leq (1/\alpha) \dim_H U$.*

Box-dimension

Definition 1.4. *$F \subset \mathbb{R}^n$ non empty, bounded. $N_\delta(F)$ the smallest number of sets of diameter at most δ which can cover F. The lower and upper box-counting dimension of F respectively defined as*

$$\underline{\dim_B}(F) = \liminf_{\delta \to 0} \frac{\log N_\delta(F)}{-log\delta} \tag{1.3}$$

$$\overline{\dim_B}(F) = \limsup_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}. \tag{1.4}$$

If these are equal the box-counting dimension is

$$\dim_B(F) = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}. \tag{1.5}$$

Proposition 1.3. *If the above limit exists it is equivalent to count with the smallest number of cubes of side δ that cover F, instead of $N_\delta(F)$.*

1.3.1 Extinction probability, Hausdorff dimension and natural measure

Extinction probability

Let $\#\mathcal{E}_n$ ($\mathcal{E}_n$ is defined in 1.1) denote the number of squares selected in the n^{th} approximation set E_n. $\{\#\mathcal{E}_n\}_{n\in\mathbb{N}}$ is a branching process, with the same offspring distribution as the distribution of $\#\mathcal{E}_1$. Hence the probability of our branching process does not die out, which is the same as the probability of E is not empty is greater than 0 if and only if $\mathbb{E}(\mathcal{E}_1) > 1$ or $\forall(i,j) \in \{0,\ldots,M-1\}^2 \quad p_{i,j} = 1$, where $\mathbb{E}(\mathcal{E}_1) = \sum\limits_{0\leq i,j\leq M-1} p_{i,j}$. In the homogeneous case the expected number of retained squares in the first level is M^2p, which means that if $p > \frac{1}{M^2}$ then $\mathbb{P}(E \neq \emptyset) > 0$.

Hausdorff dimension

The second important property of the Mandelbrot percolation set is the above defined Hausdorff dimension of it. The formula for the Hausdorff dimension is similar to the deterministic case, for self-similar sets. As Falconer [4] proved the Hausdorff dimension of a random Cantor set E is given by

$$\dim_H(E) = \frac{\log(\mathbb{E}(\#\mathcal{E}_1))}{\log M} = \frac{\log \sum\limits_{0\leq i,j\leq M-1} p_{i,j}}{\log M} \tag{1.6}$$

almost surely conditioned on $\{E \neq \emptyset\}$. Later it will be important that the Hausdorff dimension of the set E is greater than 1 if and only if $\mathbb{E}(\#\mathcal{E}_1) > M$. Again in the homogeneous case – when $\mathbb{E}(\#\mathcal{E}_1) = M^2 \cdot p$ it is straightforward that the above inequality holds if and only if $p > M^{-1}$.

Natural measure on the set E

In this section we show a method to define measures on the Mandelbrot percolation fractal. In particular we introduce two measures: the weak* limit of $\widetilde{\mu}_n$ – the normalization of the two dimensional Lebesgue measure restricted to the n^{th} approximation set, which turns out to be a probability measure, and the weak* limit of μ_n; another measure, which has a martingale property in certain cases (see Chapter 3).

Theorem 1.1 (Riesz)**.** *Let X be a locally compact Hausdorff space. For any bounded linear functional F on $C_c(X)$ there is a unique regular Borel measure μ on X such that*

$$F(f) = \int_X f(x) d\mu(x)$$

for all f in $C_c(X)$. Moreover if F is positive than μ is positive too.

Let λ_n denote the n-dimensional Lebesgue measure, and $\lambda_n|A$ the restriction of the n dimensional Lebesgue measure for the set A. For every level n approximation we can define a probability Borel measure in the following way:

$$\widetilde{\mu_n}(A) := \frac{\lambda_2|E_n(A)}{\lambda_2(E_n)} = \frac{\lambda_2(E_n \cap A)}{\#\mathcal{E}_n \cdot M^{-2n}}.$$

Now if we let n go to infinity, than we get the natural measure for the set E. From [7] we know that $\widetilde{\mu_n}$ converges in weak* sense to a measure as n goes to infinity so

$$\widetilde{\mu} = \lim_n \frac{\lambda_2|E_n}{\lambda_2(E_n)}. \tag{1.7}$$

As Mauldin and Williams([7]) use a different – more general approach I will sketch below the idea of the proof in our case. Let $W = \lim_{n\to\infty} \dfrac{\#\mathcal{E}_n}{\mathbb{E}(\#\mathcal{E}_1)^n}$.

From the theory of branching processes [1, page 9] we know that this limit exists almost surely, and greater than 0 conditioned on non extinction. Furthermore let $\mathcal{E}_{(\underline{i}_n,\underline{j}_n),k} = \{(\underline{i}_{n+k}, \underline{j}_{n+k})$: the first n terms of $\underline{i}_{n+k}$ and $\underline{j}_{n+k}$ is the fixed $\underline{i}_n$ and $\underline{j}_n$ respectively$\}$ denote the k^{th} level offsprings of $(\underline{i}_n, \underline{j}_n)$, and $W_{\underline{i}_n,\underline{j}_n} = \lim_k \frac{\#\mathcal{E}_{(\underline{i}_n,\underline{j}_n),k}}{\mathbb{E}(\#\mathcal{E}_1)^k}$, we also know that $W = \sum_{\underline{i}_n,\underline{j}_n \in \mathcal{E}_n} \frac{1}{M^{n\beta}} W_{\underline{i}_n,\underline{j}_n}$ which has the same distribution as $\frac{\#\mathcal{E}_n}{M^{n\beta}} W_{\underline{i}_n,\underline{j}_n}$ and from that $W_{\underline{i}_n,\underline{j}_n}$ has the same distribution as $\frac{M^{n\beta}}{\#\mathcal{E}_n} W$. (Note that $\mathbb{E}(\#\mathcal{E}_1)^n = (\sum_{i,j=0}^{M-1} p_{i,j})^n)$. Let β denote the Hausdorff dimension of the set E. Define a functional $F : C_c(\mathbb{R}^2) \to \mathbb{R}$ such that $F(f) = \lim_{n\to\infty} \sum_{\underline{i}_n,\underline{j}_n \in \mathcal{E}_n} f(s_{\underline{i}_n,\underline{j}_n}) \left(\frac{1}{M^n}\right)^{\beta}$ where $s_{\underline{i}_n,\underline{j}_n} \in I_{\underline{i}_n,\underline{j}_n}$. Mauldin and Williams prove that for almost all ω realization of the Mandelbrot percolation fractal and for all $f \in C_c(\mathbb{R}^2)$ F_ω is well defined positive bounded linear functional with norm $W(\omega)$. This means by Riesz theorem that there exists a regular Borel measure μ_ω on $\mathbb{R}^2$ such that $F_\omega(f) = \int_{\mathbb{R}^2} f(x) d\mu_\omega(x)$. Furthermore Mauldin and Williams prove that for all A compact subset of $\mathbb{R}^2$

$$\mu(A) = \lim_{n\to\infty} \sum_{\substack{\underline{i}_n,\underline{j}_n \in \mathcal{E}_n \\ I_{\underline{i}_n,\underline{j}_n} \cap A \neq \emptyset}} \left(\frac{1}{M^n}\right)^{\beta} W_{\underline{i}_n,\underline{j}_n} \text{ a.s..} \tag{1.8}$$

Hence

$$\mu(A) = \lim_{n\to\infty} \sum_{\substack{\underline{i}_n,\underline{j}_n \in \mathcal{E}_n \\ I_{\underline{i}_n,\underline{j}_n} \cap A \neq \emptyset}} \left(\frac{1}{M^n}\right)^{\beta} \frac{M^{n\beta}}{\#\mathcal{E}_n} W =$$

$$\lim_{n\to\infty} \sum_{\substack{\underline{i}_n,\underline{j}_n \in \mathcal{E}_n \\ I_{\underline{i}_n,\underline{j}_n} \cap A \neq \emptyset}} \frac{1}{\#\mathcal{E}_n} W = W \lim_{n\to\infty} \frac{\lambda_2 |E_n(A)}{\lambda_2(E_n)} = W \lim_{n\to\infty} \widetilde{\mu_n}(A).$$

And almost surely $\mu(E) = W$. So as $\widetilde{\mu} = \dfrac{\mu}{W}$: $\widetilde{\mu}$ is a probability measure defined on the Borel sets of $\mathbb{R}^2$.

Chapter 2

Orthogonal Projection of the set E

Among other geometrical properties, we can investigate the projection of our Mandelbrot percolation sets. First Falconer and Grimmet [5] proved that the projection to the coordinate axes contains an interval with probability 1 if $\forall i \ \sum_j p_{i,j} > 1$ and $\forall j \ \sum_i p_{i,j} > 1$ otherwise the projection almost surely does not contain any interval. Later Simon and Rams [9] showed that this can be generalized to all direction. It is straightforward that if the Hausdorff dimension of E is strictly less than one, then the Hausdorff dimension of the projection is strictly less than one by Proposition 1.2, as the projection is Hölder continuous with exponent 1. It means that the one dimensional Hausdorff measure of the set is 0, so the one dimensional outer Lebesgue measure is zero too, which means, that it can not contain any interval. In the next section I will introduce a condition which ensures almost sure non-empty interior for the projection to all directions. We will not cover the

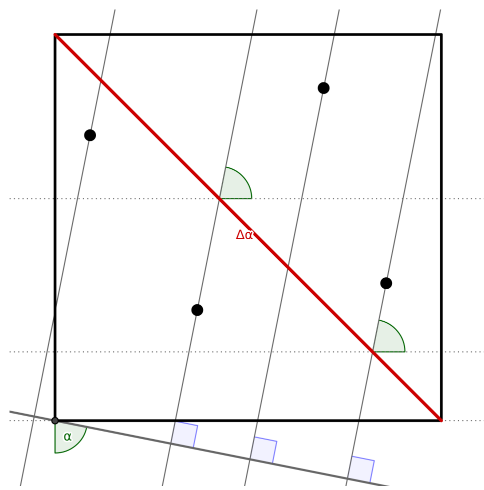

Figure 2.1: The modified projection

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whole proof only the intuition behind the proof.

2.1 Condition A(α) and proof

For the above mentioned reason assume that the Hausdorff dimension of E is strictly greater than 1, so $\sum_{i,j=0}^{M-1} p_{i,j} > M$. Instead of looking at the orthogonal projection to a line which has α angle to the x-axis, we will investigate the non-orthogonal projection to one of the diagonals of E – denoted with Δ_α – depending on the size of α. If $\alpha \in \left(0, \frac{\pi}{2}\right)$ then Δ_α is the interval $([0,0],[1,1])$, and if $\alpha \in (\frac{\pi}{2}, \pi)$ then Δ_α is $([0,1],[1,0])$. Denote this projection with $\Pi_\alpha^{\Delta_\alpha} : I \to \Delta_\alpha$. Also let $\mathscr{L}_\alpha(x) = (\Pi_\alpha^{\Delta_\alpha})^{-1}(x)$ denote the segment through $x \in \Delta_\alpha$ with angle α in I (see Figure 2.1). Let $\phi_{\underline{i}_n,\underline{j}_n} : I \to I_{\underline{i}_n,\underline{j}_n}$ denote the contraction, namely $\phi_{\underline{i}_n,\underline{j}_n}(x,y) := M^{-n}(x,y) + t_{\underline{i}_n,\underline{j}_n}$, where $t_{\underline{i}_n,\underline{j}_n}$ is the lower left corner of $I_{\underline{i}_n,\underline{j}_n}$. Simon and Rams define Condition

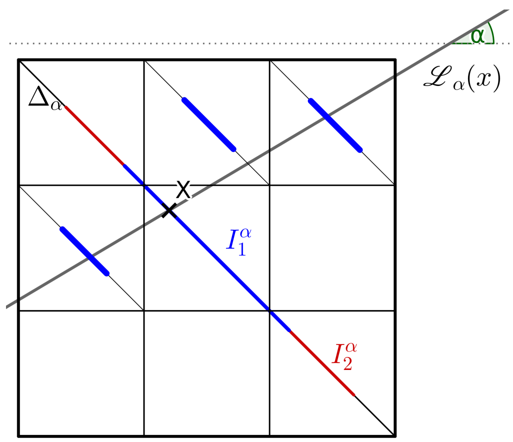
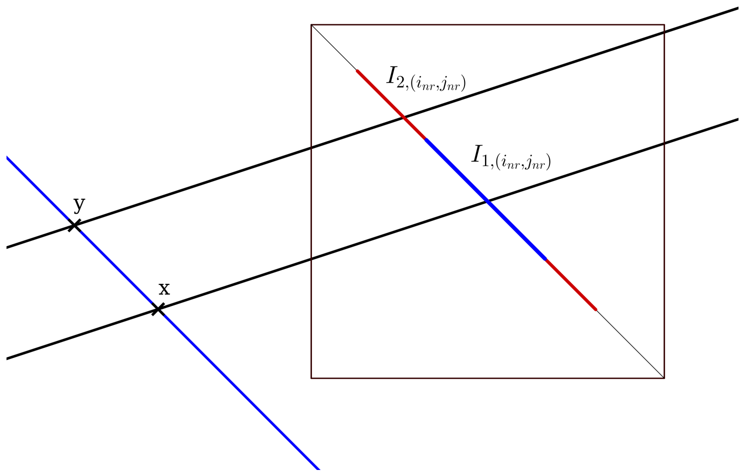


Figure 2.2: $\mathscr{L}_\alpha(x)$, I_1^α, I_2^α.

A(α): for a given α the percolation model satisfies it, if there exist closed intervals I_1^α, I_2^α in the interior of Δ_α, and a positive integer r_α such that I_2^α is in the interior of I_2^α and for all $x \in I_2^\alpha$ the expected number of r_α level small squares which intersects $\mathscr{L}_\alpha(x)$ in a point which contained in the $\phi_{\underline{i}_{r_\alpha},\underline{j}_{r_\alpha}}(I_1^\alpha)$ interval is greater or equal than 2 (see Figure 2.2).

If that condition is satisfied for a given α then the projection contains an interval with positive probability. For the proof Simon and Rams use large deviation estimation, the robustness of Condition A(α) and the statistical self similarity of the set. The robustness of A(α) means that there is small neighborhood of α such that if Condition A(α) holds, then it holds in that small neighborhood too, and in that neighborhood we can use the same I_1, I_2 intervals and r_α integer. To see this choose δ to be the greatest number such that the δ-neighborhood of I_1 is still in the interior of I_2, as these are closed δ is positive. Let I_3 denote the closure of the $\frac{\delta}{2}$ neighborhood of I_1. Now if $\tau \in [\alpha - \frac{\delta}{3M^{r_\alpha}}, \alpha + \frac{\delta}{3M^{r_\alpha}}]$ then the maximum distance between $\mathscr{L}_\alpha(x)$



and $\mathscr{L}_\tau(x)$ in $I \times I$ is less than $\frac{\delta}{2M^{r\alpha}}$ which means that those small squares which $\mathscr{L}_\alpha(x)$ intersects in a point in $\phi_{\underline{i}_r,\underline{j}_r}(I_1)$, intersects $\mathscr{L}_\tau(x)$ in a point in $\phi_{\underline{i}_r,\underline{j}_r}(I_3)$, call it intersection in the right way. The extract of the proof is the following. First we show that if Condition A(α) hold for a given α then the interior of the projection will not be empty, thus we can conclude that if $H \subset (0, pi/2)$ and for all $\alpha \in H$ Condition A(α) holds then for all $\alpha \in H$ for almost all realization $\text{proj}_\alpha E$ contains an interval. This means that for every $\alpha \in H$ the set of bad realizations has measure 0. On the second part we show that we can state something stronger (see 2.1) namely that if H has certain properties then the union for $\alpha \in H$ of bad realizations has measure 0 even if H is an uncountable set. Assume that Condition A(α) holds with I_1, I_2, r. Let $D_n(x, I, \alpha) = \{(\underline{i}_n, \underline{j}_n) : \mathscr{L}_\alpha(x) \text{ intersects } \phi_{\underline{i}_n,\underline{j}_n}(I)\}$, and $V_n(x) = \#\{(\underline{i}_{nr}, \underline{j}_{nr}) \in \mathcal{E}_{nr} \cap D_{nr}(x, I_1, \alpha)\}$ the number of level $n \cdot r$ squares we kept, which intersects $\mathscr{L}_\alpha(x)$ in the right way. We will show that I_1 is contained in the projection with positive probability, so for all $x \in I_1$ at every level we can find squares which we kept, and the projection of the square contains x, more precisely – to be able to use statistical self similarity – the projection contains x in the middle. For that define a finite set X_n such that X_n contains the endpoints of I_1 and the distance between its points is less or equal than δM^{-nr}, where $\delta = \sup\{d : B_d(I_1) \subset I_2\}$. in that way the size of X_n will be relatively small ($\#X_n \leq c\delta M^{-nr}$), and the $I_{\underline{i}_{nr},\underline{j}_{nr}}$ squares which intersects $\mathscr{L}_\alpha(x_m)$ in $\phi_{\underline{i}_{nr},\underline{j}_{nr}}(I_1)$ will intersect $\mathscr{L}_\alpha(x_{m-1})$ and $\mathscr{L}_\alpha(x_{m+1})$ in $\phi_{\underline{i}_{nr},\underline{j}_{nr}}(I_2)$ (So $\forall y \in [x_{m-1}, x_{m+1}] : D_{nr}(x_m, I_1, \alpha) \subset D_{nr}(y, I_2, \alpha)$) By Condition A($\alpha$) this means that the expected number of level $(n+1)r$ level squares which intersects $\mathscr{L}_\alpha(x_{m\pm1})$ in the right way is at least twice as much as those which intersects $\mathscr{L}_\alpha(x_m)$ in the almost right way – in some $\phi_{\underline{i}_{nr},\underline{j}_{nr}}(I_2)$. Using this if $V_n(x_n) \geq ({}^3/_2)^n \ \forall x_n \in X_n$ than for each level $n \cdot r$ square which we counted, the expected number of offsprings is greater or equal than 2. As ${}^3/_2 < 2$ and the number of squares is greater or equal than $({}^3/_2)^n$ using the Chernoff bound we get the following for all $x_{n+1} \in X_{n+1}$:

$$\mathbb{P}\big(V_{n+1}(x_{n+1}) < (3/2)^{n+1} | \forall x_n \in X_n \ V_n(x_n) \geq (3/2)^n\big) \leq e^{-(3/2)^n I(3/2)}$$

hence there exist a $0 < \Gamma < 1$ such that:

$$\mathbb{P}\big(V_{n+1}(x_{n+1}) < (3/2)^{n+1} | \forall x_n \in X_n \ V_n(x_n) \geq (3/2)^n\big) \leq \Gamma^{(3/2)^n}$$

thus:

$$\mathbb{P}\big(V_{n+1}(x_{n+1}) \geq (3/2)^{n+1} | \forall x_n \in X_n \ V_n(x_n) \geq (3/2)^n\big) < (1 - \Gamma^{(3/2)^n})$$

which means using conditional independence and that $\#X_{n+1} \leq cM^{(n+1)r}$:

$$\mathbb{P}\big(\forall x_{n+1} \in X_{n+1} : V_{n+1}(x_{n+1}) \geq (3/2)^{n+1} | \forall x_n \in X_n : V_n(x_n) \geq (3/2)^n\big) < \big(1 - \Gamma^{(3/2)^n}\big)^{cM^{(n+1)r}}.$$

As $V_0(x) \geq ({}^3/{}_2)^0$

$$\mathbb{P}\big(\forall n \forall x \in X_n : V_n(x) \geq (3/2)^n\big) \geq \mathbb{P}(V_0 \geq 1) \prod_n \big(1 - \Gamma^{(3/2)^n}\big)^{cM^{(n+1)r}} > 0.$$

The last inequality holds, because the product converges to non zero number if and only if the sum $\sum_n \log\big[(1-\Gamma^{3/2^n})^{cM^{(n+1)r}}\big] = \sum_n cM^{(n+1)r} \log\big(1 - \Gamma^{(3/2)^n}\big) \leq \sum_n cM^{(n+1)r}\Gamma^{(3/2)^n}$ converges, and it does. With this we are ready, because if x is contained in I_1 but $\exists n$ such that $x \notin X_n$, at every $n \cdot r$ we can find $x_n \in X_n$ such that x_n is close enough to x and those level $n \cdot r$ squares which project to x_n project to x too as $D_{nr}(x_n, I_1, \alpha) \subset D_{nr}(x, I_2, \alpha)$. Using statistical self similarity this means that for all n for all level n small square in E_n the probability that their projection does contain an interval is $\varepsilon > 0$. So $\mathbb{P}$(the projection of E contains no interval conditioned on E is not empty) $\leq \mathbb{P}(\#\mathcal{E}_n < N | E \neq \emptyset) + (1-\varepsilon)^N$. The first part tends to 0 as $n \to \infty$ and then as $N \to \infty$ the expression tends to 0. For set of angles the proof is modified in a way that we estimate the probability of the unwanted event not just for a finite set of points in Δ_α but for a finite set of directions too, this again is a relatively small set compared to the super-exponentially small probability of the unwanted event. For a similar proof see Chapter 3.2.2. For the final result consider a compact set K of angles. By the robustness of Condition A for every $\alpha \in K$ there exists an interval where Condition A(α) is satisfied. We can choose this interval to have rational endpoints (call this interval J_α), and a set of these makes a countable cover of K, so as K is compact, there is a finite cover of K with the sets $\{J_{\alpha_i,i}\}_{i=1}^k$. For all i for almost all realization for all $\tau \in J_{\alpha_i,i}$ the projection contains an interval. This means that for all $i \in \{1, 2, \ldots, k\}$ the set of bad realizations (which projection has empty interior) has measure zero, so the finite union of these realizations is a zero measure set too, and we are done.

Theorem 2.1. *If $K \in [0, 2\pi]$ a compact set of angles, and $\forall \alpha \in K$ Condition $A(\alpha)$ is satisfied, than for almost all realizations for all $\alpha \in K$ $proj_\alpha(E)$ contains an interval.*

2.2 Condition B(α)

In most cases Condition A(α) is not easy to check, so Simon and Rams introduce Condition B(α) which implies Condition A(α), and which can easily be checked for example in the homogeneous case. For this we introduce some more notation, to be able to define Condition A(α) more precisely. First recall, that $\Pi_\alpha^{\Delta_\alpha}$ is the projection onto the diagonal of I: Δ_α and $\phi_{\underline{i}_n,\underline{j}_n}$ is the contraction of I to $I_{\underline{i}_n,\underline{j}_n}$. Let $\psi_{\alpha,\underline{i}_n,\underline{j}_n}$ denote the inverse of $\Pi_\alpha^{\Delta_\alpha} \circ \phi_{\underline{i}_n,\underline{j}_n} : \Delta_\alpha \to \Delta_\alpha$.

$$F_\alpha f(x) = \sum_{\substack{(i,j) \\ x \in \Pi_\alpha^{\Delta_\alpha}(I_{i,j})}} p_{i,j} \cdot f \circ \psi_{\alpha,i,j}(x) \tag{2.1}$$

and similarly

$$F_\alpha^n f(x) = \sum_{\substack{(\underline{i}_n,\underline{j}_n) \\ x \in \Pi_\alpha^{\Delta_\alpha}(I_{\underline{i}_n,\underline{j}_n})}} p_{\underline{i}_n,\underline{j}_n} \cdot f \circ \psi_{\alpha,\underline{i}_n,\underline{j}_n}(x) \tag{2.2}$$

Definition 2.1 (Condition A(α)). *The fractal percolation model satisfies condition A(α) if there exist closed intervals $I_1^\alpha, I_2^\alpha \subset \Delta_\alpha$ and a positive integer r_α such that:*

(i) $I_1^\alpha \subset$ *int* I_2^α*;*

(ii) $I_2^\alpha \subset$ *int* Δ_α*;*

(iii) $F_\alpha^{r_\alpha} \mathbb{1}_{I_1^\alpha} \geq 2\mathbb{1}_{I_2^\alpha}$.

Definition 2.2 (Condition B(α)). *A fractal percolation model satisfies Condition B(α) if there exist a continuous function $f : \Delta_\alpha \to \mathbb{R}$ such that f is strictly positive except at the endpoints of Δ_α and $F_\alpha f \geq (1+\varepsilon) f$ for some $\varepsilon > 0$.*

Proposition 2.1. *Condition B(α) implies Condition A(α).*

Proof. Assume that Condition B(α) holds for some f and ε. In the first part of the proof I will show that we can choose non-empty closed intervals $I_1 \subset \mathrm{int} I_2$ and $I_2 \subset \mathrm{int}\Delta_\alpha$ such that for

$$g_1 = f|_{I_1}, g_2 = f|_{I_2} : F_\alpha g_1(x) \geq (1+\varepsilon/2) \cdot g_2(x) \ \forall x \in I_2. \tag{2.3}$$

Let $W := \{x \in \Delta_\alpha : \exists \ 0 \geq i, j, \geq M, \ x = \Pi_\alpha^{\Delta_\alpha}(i/M, j/M)\}$ the projection of the mesh $1/M$ grid points in I. Let W_0 denote the two endpoints of Δ_α, and $W_1 = W \setminus W_0$. And let $\eta > 0$ be fixed such that:

$$\frac{\varepsilon}{2} \cdot \min_{x \in B_{\eta/M}(W_1)} f(x) > (M+1)^2 \sup_x \{f(x) : x \in B_\eta(W_0)\}. \tag{2.4}$$

If we let I_1 be $\Delta_\alpha \setminus B_\eta(W_0)$ and I_2 to be $\Delta_\alpha \setminus B_{\eta/M}(W_0)$ then equation 2.3 holds, because $I_2 = (\Delta_\alpha \setminus B_{\eta/M}(W)) \cup B_{\eta/M}(W_1)$.

1. If $x \in \Delta_\alpha \setminus B_{\eta/M}(W)$ then $F_\alpha g_1(x) = F_\alpha f(x) \geq (1+\varepsilon)f(x) \geq (1+\varepsilon/2)g_2(x)$.

2. And if $x \in B_{\eta/M}(W_1)$ then $F_\alpha g_1(x) \geq F_\alpha f(x) - (M+1)^2\|f - g_1\|_\infty \geq (1+\varepsilon/2)f(x) + (\varepsilon/2 f(x) - (M+1)^2\|f-g_1\|_\infty)$ and as the second part is greater than 0 by the definition of η: $F_\alpha g_1(x) > (1+\varepsilon/2)g_2(x)$.

Now let r be the smallest integer satisfying

$$\left(1+\frac{\varepsilon}{2}\right)^r \geq 2 \cdot \frac{\max_{x\in I_1} g_1(x)}{\min_{x \in I_2} g_2(x)}$$

with this choice of r:

$$F_\alpha^r \mathbb{1}_{I_1}(x) \geq F_\alpha^r \frac{g_1(x)}{\max_{x\in I_1} g_1(x)} = \frac{1}{\max_{x\in I_1} g_1(x)} F_\alpha^r g_1(x) \geq \frac{(1+\varepsilon/2)^r}{\max_{x\in I_1} g1(x)} g_2(x) \geq 2 \frac{g_2(x)}{\min_{x\in I_2} g_2(x)} \geq 2\mathbb{1}_{I_2}(x) \text{ for all } x \in I_2.$$

□

2.2.1 Homogeneous case

Proposition 2.2 (Non empty interior in the homogeneous case)**.** *If* $\forall (i,j) \in \{1,\dots,M-1\}^2$ $p_{i,j} = p$ *and* $\dim_H E > 1$ *or equivalently* $p > 1/M$ *then Condition A(α) is satisfied for all* $\alpha \in (0, 2\pi)$.

Proof. Let $f_\alpha(x) = |\mathscr{L}_\alpha(x)|$, the length of the line segment through x in with angle α. f is obviously continuous, and

$$F_\alpha f(x) = p \sum_{(i,j)} M \cdot |\mathscr{L}_\alpha \cap I_{i,j}| = M \cdot p \cdot f(x) \geq (1+\varepsilon) f(x)$$

as $M \cdot p > 1$. Which means Condition B(α) is satisfied and so Condition A(α). □

Using this and the above mentioned result of Falconer and Grimmet we can conclude that in the homogeneous case the projection almost surely contains an interval for all angle.

2.3 Hausdorff dimension and empty interior

At this point we can ask the question, whether it is true that Mandelbrot percolation fractals with Hausdorff dimension greater than 1 has non-empty interior for almost all realization for all angle. The answer is yes in the homogeneous case, but in general no if the probabilities are not the same. Let me show a family of counterexamples.

Example

Let Σ be a subset of $\{0,1,\ldots,M-1\}\times\{0,1,\ldots,M-1\}$ and $\#\Sigma > M$ denote Λ the attractor of $\Psi = \{F_\omega\}_{\omega\in\Sigma}$, where $F_{k,l}(x,y) = \frac{1}{M}(x,y)+\frac{1}{M}(k,l)$. From [2] we know that

Proposition 2.3. *$M \nmid \#\Sigma$, than for every fixed $\tau \in [0,\pi/2)$ such that $\tan\tau \in \mathbb{Q}$ and $a \in \Pi_\tau^y\Lambda = [-\tan\tau, 1]$*

$$\dim_B \mathscr{L}_\tau(a) < \frac{\log \#\Sigma}{\log M} - 1 \text{ for Lebesgue almost all } a \in \Pi_\tau^y. \tag{2.5}$$

Where $\Pi_\tau^y(x,y) = y - x\tan\tau$ the projection to the y-axis with angle τ and $\mathscr{L}_\tau(a)$ is the intersection of the line through a with angle τ and Λ. Construct E in the way that $M/\#\Sigma < 1$ and $\forall (i,j) \in \Sigma$ $p_{i,j} = p$ and otherwise $p_{i,j} = 0$, where $p > \frac{M}{\#\Sigma}$ i.e. $\dim_H E > 1$. Following the proof in [11] I will show that for any $\tau \in [0,\pi/2)$ such that $\tan\tau \in \mathbb{Q}$ $\exists p_\tau > \frac{M}{\#\Sigma}$

such that for $\frac{M}{\#\Sigma} < p < p_\tau$ the projected percolation set $\Pi_\tau^y(E)$ has empty interior almost surely. By Proposition 2.3 for Lebesgue almost all $a \in \Pi_\tau^y(I)$ $\dim_B \mathscr{L}_\tau(a) = \frac{\log((\#\Sigma/M)(1-\varepsilon_\tau))}{\log M}$ for $\varepsilon_\tau > 0$. Let $\Lambda_{\tau,a,n} = \{(i_n, j_n) \in \Sigma : \mathscr{L}_\tau(a) \cap I_{i_n,j_n} \cap \Lambda \neq \emptyset\}$ and $N_{\tau,a,n} = \#\Lambda_{\tau,a,n}$ and $\mathcal{E}_{\tau,a,n} = \mathcal{E}_n \cap \Lambda_{\tau,a,n}$. As $\frac{\log N_{\tau,a,n}}{\log M^n}$ converges to the box dimension of the set $\exists\widetilde{n}$ depending on τ, a s.t. $\forall n > \widetilde{n}$

$$\frac{\log N_{\tau,a,n}}{\log M^n} < \frac{\log(\#\Sigma/M(1-\varepsilon_\tau)(1+\epsilon_\tau))}{\log M}$$

$$N_{\tau,a,n} < \big(\frac{\#\Sigma}{M}\big)^n(1-\varepsilon_\tau)^n$$

$$\mathbb{E}(\#\mathcal{E}_{\tau,a,n}) < N_{\tau,a,n}p^n$$

the last expression tends to zero if $p < \frac{M}{\#\Sigma}\frac{1}{(1-\varepsilon_\tau)^2}$. From the Markov inequality for all $\delta > 0$:

$$\mathbb{P}(\#\mathcal{E}_{\tau,a,n} > \delta) < \frac{\mathbb{E}(\#\mathcal{E}_{\tau,a,n})}{\delta} < \frac{N_{\tau,a,n}p^n}{\delta} \to 0 \text{ as n tends to } \infty$$

which means that the number of squares project to a for almost all a in $\Pi_\tau^y I$ tends to 0, so the interior will be empty.

Chapter 3

Projection of the natural measure

3.1 Projection of a measure

Recall that the natural measure on E_n is $\widetilde{\mu_n} = \dfrac{\lambda_2|E_n}{\lambda_2(E_n)}$ and on E $\widetilde{\mu} = \lim_n \dfrac{\lambda_2|E_n}{\lambda_2(E_n)}$ regular measures on the Borel sets of $\mathbb{R}^2$ supported on E_n and E respectively. It is possible to define the projection of measures, as the push forward measure by the measurable function – with respect to the Borel sigma algebra – $\text{proj}_\alpha : I \to \text{proj}_\alpha(I)$. Where $\text{proj}_\alpha(x, y) = x\cos(\alpha) + y\sin(\alpha)$. The projected measure will be the following: $\forall A \in \mathcal{B}(\text{proj}_\alpha(I))$:

$$\text{proj}^*_\alpha\mu(A) = \mu(\text{proj}^{-1}_\alpha(A))$$

Denote the projected measure with μ^α. As μ and μ_n are Radon or regular measures μ_α and μ^α_n are Radon measures too (A proof can be found in [6, page 16]). Our goal in this section is to show cases when this projected measure is absolute continuous with respect to the Lebesgue measure, moreover to show that the density is Hölder continuous.

3.2 Homogeneous case

As I mentioned above the Mandelbrot percolation set with homogeneous probabilities has a.s. not empty interior in the case when $\dim_H E > 1$. And as Peres and Rams show the projected measure is also absolutely continuous with respect to the Lebesgue measure and also Hölder continuous in every direction except the vertical and horizontal one. More precisely Peres and Rams prove the following theorem.

Theorem 3.1. *Assume $Mp > 1$. If E is non-empty then almost surely all the projections $\mu^\alpha = proj_\alpha^* \mu$ are absolutely continuous with respect to the Lebesgue measure. Moreover, almost surely the density of μ^α is Hölder continuous for $\alpha \neq 0, \pi/2$. For the horizontal and vertical projections the density of the projected measure will in general be undefined at the M-adic points, but it will almost surely be Hölder continuous in the metric*

$$\rho(x, y) = \exp(-\log M \cdot min\{\ell : \exists m : x < mM^{-\ell} < y\})$$

everywhere except at the M-adic points.

Recall that $W = \lim_{n\to\infty} \mathbb{E}(\#\mathcal{E}_1)^{-n}\#\mathcal{E}_n$, in the homogeneous case $W = \lim_{n\to\infty}(M^2p)^{-n}\#\mathcal{E}_n$. In that case $\widetilde{\mu} = \lim_{n\to\infty} \frac{\lambda_2|E_n}{p^n W}$. In our case it is better to use $\mu_n = \frac{\lambda_2|E_n}{p^n}$ which converges to $\mu = W\widetilde{\mu}$, because it has an important property which has a key role in proving absolute continuity, namely that it is a martingale with respect to the sigma algebra generated by the survival set $\mathcal{E}_n$:

$$\mathbb{E}(\mu_{n+1} \mid \mathcal{E}_n) = \mu_n. \tag{3.1}$$

To see that it is true consider the following:

$$\mathbb{E}(\mu_{n+1} \mid \mathcal{E}_n) = p^{-n-1} \sum_{(i_n,j_n)\in\mathcal{E}_n} \sum_{k=1}^{M^2} p \cdot \frac{\lambda_2|I_{i_n,j_n}}{M^2} = p^{-n-1} \sum_{(i_n,j_n)\in\mathcal{E}_n} p \cdot \lambda_2|I_{i_n,j_n} = \frac{\lambda_2|E_n}{p^n}$$

Note that this does not hold in general when the probabilities are not the same. Now focus on the projected measure μ_n^α. As it is absolutely continuous with respect to the Lebesgue measure it has a density function y_n^α, which also has a martingale property. The first part of the proof is an estimation of $y_{n+1}^\alpha(x)$ with $y_n^\alpha(x)$.

$$y_n^\alpha(x) = \frac{\left|\mathscr{L}_\alpha(x) \cap E_n\right|}{p^n} \tag{3.2}$$

Where $\mathscr{L}_\alpha(x)$ is $\text{proj}_\alpha^{-1}(x)$ the line through x with angle α. Define a random variable, the length of the intersection of the line segment $\mathscr{L}_\alpha(x)$ and $I_{\underline{i}_n i,\underline{j}_n j}$ if $\underline{i}_n i, \underline{j}_n j \in \mathcal{E}_{n+1}$:

$$Y(\underline{i}_n, \underline{j}_n; x; \alpha) := \left|\mathscr{L}_\alpha(x) \cap I_{\underline{i}_n,\underline{j}_n} \cap E_{n+1}\right|. \tag{3.3}$$

$$y_{n+1}^\alpha(x) = \frac{1}{p^{-n-1}} \sum_{(\underline{i}_n,\underline{j}_n)\in\mathcal{E}_n} Y(\underline{i}_n, \underline{j}_n; x, \alpha) \tag{3.4}$$

Proposition 3.1. *Let X_i be a family of independent bounded random variables with $\mathbb{E}(X_i)$=0 and $\|X_i\| = sup_\omega|X_i|(\omega) \leq 1$. If $S = \sum X_i$ and $\Gamma = \sum\|X_i\|$ then for all $a > 0$:*

$$\mathbb{P}(S > a) \leq \exp(-a^2/2\Gamma) \tag{3.5}$$

Proposition 3.2. *There exist* $C_1 > 0$ *and* $\gamma < 1$ *such that the following statements are true.*

(i) If $x, \alpha, \mathcal{E}_n$ *satisfy* $y_n^\alpha(x) > 1$ *then*

$$\mathbb{P}\big(y_{n+1}^\alpha(x) < y_n^\alpha(x) + p^{-n}M^{-n}(p^n M^n y_n^\alpha(x))^{2/3} | \mathcal{E}_n\big) > 1 - C_1 \gamma^{(pM)^{n/3}}$$

(ii) If $x, \alpha, \mathcal{E}_n$ *satisfy* $y_n^\alpha(x) < (pM)^{n/3}$ *then*

$$\mathbb{P}\big(|y_{n+1}^\alpha(x) - y_n^\alpha(x)| < (pM)^{-n/6} | \mathcal{E}_n\big) > 1 - C_1 \gamma^{(pM)^{n/3}}$$

Proof. For the first part: if we choose $X_{\underline{i}_n,\underline{j}_n} = M^n(Y(\underline{i}_n, \underline{j}_n; x, \alpha) - p|\mathscr{L}_x^\alpha \cap I_{\underline{i}_n,\underline{j}_n}|)/\sqrt{2}$ and $a = 1/\sqrt{2}p(p^n M^n y_n^\alpha(x))^{2/3}$ the assertion follows from Proposition 3.1. And for the second part choose $a = 1/\sqrt{2}p(pM)^{5n/6}$. For more details see [8, page 544] □

3.2.1 Horizontal and vertical projections

The case of the horizontal and the vertical projection is the same by symmetrical reasons, so consider the vertical projection. Let $K_{\underline{i}_n}$ denote the M-adic interval with length M^{-n} and with index $\underline{i}_n$. In the vertical case $y_n^{\pi/2}$ – which in this section I will denote with y_n – is constant on the M-adic intervals of level n, so let $y_n'(\underline{i}_n) = y_n(x)$ if $x \in K_{\underline{i}_n}$.

Let N_0 be the smallest number for which

$$1 + (pM)^{-N_0/3} < (pM)^{1/8} \tag{3.6}$$

holds. This N_0 surely exists, as $pM > 1$, so for a large N_0 $(pM)^{-N_0/3}$ is close to 0 and also $(pM)^{1/8}$ is still greater than 1. As $1 + (pM)^{-5N_0/3} <$

$1+(pM)^{-N_0/3}<(pM)^{1/8}<(pM)^{1/4}$:

$$1+(pM)^{-5N_0/3}<(pM)^{1/4} \tag{3.7}$$

also holds.

Proposition 3.3. *If $N>N_0$ and $y'_N(\underline{i}_N)<(pM)^{N/4}$ than for all $x\in K_{\underline{i}_N}$*

$$\mathbb{P}\big(\forall n\geq N\ |y_{n+1}(x)-y_n(x)|<(pM)^{-n/6}|\mathcal{E}_N\big)\geq 1-C_1\sum_{m=N}^{\infty}\gamma^{(pM)^{m/3}} \tag{3.8}$$

and also

$$\mathbb{P}\big(\forall n\geq N\ \forall x\in K_{i_n}\ |y_{n+1}(x)-y_n(x)|<(pM)^{-n/6}|\mathcal{E}_N\big)\geq$$
$$\geq 1-C1\sum_{m=N}^{\infty}M^{m-N}\gamma^{(pM)^{m/3}}. \tag{3.9}$$

Proof. From the second part of Proposition 3.2 we know that if $y_N(x)<(pM)^{N/3}$ then $\mathbb{P}(|y^{\alpha}_{N+1}(x)-y^{\alpha}_N(x)|>(pM)^{-N/6}|\mathcal{E}_N)<C_1\gamma^{(pM)^{N/3}}$. Let Q_k be the event that the event above happens for all n up to k, namely

$$Q_k=\{n=N,\dots,k\ |y_{n+1}(x)-y_n(x)|<(pM)^{-n/6}\}$$

and Q_∞ the event in the proposition.

$$\mathbb{P}\big(Q^c_\infty|\mathcal{E}_N\big)=\mathbb{P}\big(Q^c_N|\mathcal{E}_N\big)+\mathbb{P}\big(Q_N\cap Q^c_{N+1}|\mathcal{E}_N\big)+$$
$$+\mathbb{P}\big(Q_N\cap Q_{N+1}\cap Q^c_{N+2}|\mathcal{E}_N\big)+\cdots<\mathbb{P}\big(|y_{N+1}(x)-y_N(x)|>(pM)^{-N/6}\big)+$$
$$+\mathbb{P}\big(|y_{N+2}(x)-y_{N+1}(x)|>(pM)^{-(N+1)/6}|Q_N\cap\mathcal{E}_N\big)+\cdots<C_1\sum_{m=N}^{\infty}\gamma^{(pM)^{m/3}}$$

The last inequality holds because we can use Proposition 3.2 as if Q_k happens, then for $k+1$ $y_{k+1}(\underline{i}_k i)<(pM)^{(k+1)/4}$ as $|y_{k+1}(x)-y_k(x)|<(pM)^{-k/6}$ so

$y_{k+1}(x) < y_k(x) + (pM)^{-k/6} < (pM)^{k/4} + (pM)^{-k/6} < (pM)^{(k+1)/4}$. The last inequality holds because $k \geq N_0$ so $1 + (pM)^{-5k/12} < (pM)^{1/4}$, which multiplied with $(pM)^{k/4}$ gives the inequality. The proof of the second part is similar, only it has to happen for all n and for all M^{n-N} sequences $\underline{i}_n$ beginning with $\underline{i}_N$. □

Proposition 3.4. *There exist an $L > 1$ such that for all $n > LN_0$ and for all x:*

$$\mathbb{P}\big(y_n(x) < (pM)^{n/4}\big) > 1 - C_1 \sum_{m=n/L}^{n} \gamma^{(pM)^{m/3}}.$$

Proof. There will be three time periods. The first period, when $m \in [0, N_0]$, will be out of our interest. For the second period, $m \in [N_0, l_0]$ $y_m(x)$ might be large, but we will show that $1/m \log_{pM} y_m(x)$ will be decreasing, and eventually decreasing below 1/4. For the third period, when $m \geq l_0$ $y_m(x) < (pM)^{m/4}$ and thus we can apply Proposition 3.3.

Start with the second period: if $y_{N_0} \leq (pM)^{N_0/4}$ then the second period does not exist and we can jump to the third period immediately. If $y_{N_0} > (pM)^{N_0/4}$ then the second period exist, and as long as $y_m(x) > 1$ by Proposition 3.2, and Equation 3.6:

$$\begin{aligned} y_m(x) + (pM)^{-m}((pM)^m y_m(x))^{2/3} &= y_m(x)(1 + (pM)^{-m/3} y_m(x)^{-1/3}) \\ &\leq y_m(x)(pM)^{1/8} \end{aligned}$$

so by Proposition 3.2:

$$\begin{aligned} &\mathbb{P}\big(\log_{pM} y_{m+1}(x) < \log_{pM} y_m(x) + 1/8\big) = \mathbb{P}\big(y_{m+1}(x) < y_m(x)(pM)^{1/8}\big) \\ &\quad \geq \mathbb{P}\big(y_{m+1}(x) < y_m(x) + (pM)^{-m}((pM)^m y_m(x))^{2/3}\big) \geq 1 - C_1 \gamma^{(pM)^{m/3}} \end{aligned}$$

hence $\log_{pM} y_{m+1}(x) < \log_{pM} y_m(x) + \frac{1}{8}$ with high probability. Thus, if the event in Propoisition 3.1 holds for each $m \geq N_0$ then:

$$y_{lN_0}(x) \leq y_{lN_0-1}(x)(pM)^{1/8} \leq \cdots \leq y_{N_0}(pM)^{N_0(l-1)/8}$$

$|\mathscr{L}_{\pi/2}(x) \cap E_{N_0}| \leq 1$ hence $y_{N_0}(x) \leq p^{-N_0}$

$$y_{lN_0}(x) \leq (pM)^{N_0(l-1)/8} p^{-N_0}.$$

This means that $\log_{pM} y_m(x) < (pM)^{m/8} - N_0 \log_{pk} p$ therefore eventually $1/m \log_{pM} y_m(x)$ will be less than $1/4$ (if the events in Proposition 3.2 ii happens for all $m \geq N_0$). Let l_0 be the smallest number for which this happens ($y_{l_0}(x) < (pM)^{l_0/4}$), and define L:

$$L = \lceil -8 \log_{pM} p \rceil + 1$$

As $(pM)^{LN_0/4} \geq p^{-2N_0}(pM)^{N_0/4} > p^{-2N_0} = (pM)^{(L-1)N_0/8} p^{-N_0}$ $l_0 \leq LN_0$. From the proof of Proposition 3.2 if $N > N_0$ and $y_N(x) < (pM)^{N/4}$ then for any $n \geq N$

$$\mathbb{P}(N \leq m \leq n \; |y_{m+1}(x) - y_m(x)| < (pM)^{-m/6} | \mathcal{E}_N) \geq 1 - C_1 \sum_{m=N}^{n} \gamma^{(pM)^{m/3}}. \tag{3.10}$$

The assertion holds for $n \geq LN_0$ if for all $n \geq m \geq N_0$ the event in Proposition 3.2 ii happens, and the event in equation 3.10 holds with $N = l_0$, but if the first event happens up to l_0 and the second from l_0 then the first happens from l_0. From this

$$\begin{aligned}
&\mathbb{P}(y_n(x) < (pM)^{n/4}) \\
&\geq \max_{l_0}\left[\left(1 - C_1 \sum_{m=N_0}^{l_0-1} \gamma^{(pM)^{m/3}}\right)\left(1 - C_1 \sum_{m=l_0}^{n} \gamma^{(pM)^{m/3}}\right)\right] \\
&> 1 - C_1 \sum_{m=n/L}^{n} \gamma^{(pM)^{m/3}}
\end{aligned}$$

Using Propositions 3.3 and 3.4 we can prove the last Proposition of this section, which leads us to the main result of this section. □

Proposition 3.5. *There exists* $b < 1$ *such that almost surely there exist* $C_2 > 0$ *such that for all* $x \in [0, 1]$ *except the M-adic points and for all* $N > LN_0$ *we have*

$$\left| y_N(x) - \lim_{n\to\infty} y_n(x) \right| < C_2 b^N$$

Proof. If $N > LN_0$ then for all $\underline{i}_N$ for all $x \in K_{\underline{i}_n}$ for all $m \geq N$

$$\begin{aligned}
&\mathbb{P}\left(|y_N(x) - y_m(x)| > \sum_{n=N}^{m} (pM)^{-n/6} \right) \\
&\leq \mathbb{P}\left(\sum_{n=N}^{m} |y_n(x) - y_{n+1}(x)| > \sum_{n=N}^{m} (pM)^{-n/6} \right) \\
&\leq \mathbb{P}\big(y_N(x) > (pM)^{N/4}\big) + \\
&\sum_{n=N}^{m} \mathbb{P}(\forall x \in K_{i_N} |y_n(x) - y_{n+1}(x)| > (pM)^{-n/6} \mid \forall x \in K_{i_N}\ y_n(x) < (pM)^{n/4}) \\
&\leq C_1 \sum_{n=N/L}^{N} \gamma^{(pM)^{n/3}} + C_1 \sum_{n=N}^{m} M^{N-n} \gamma^{(pM)^{n/3}}
\end{aligned}$$

If we let $m \to \infty$, and taking the complement event then we get:

$$\mathbb{P}\left(\lim_{n\to\infty} y_n(x) \text{ exists and } |y_N(x) - \lim_{n\to\infty} y_n(x)| < \frac{1}{1-(pM)^{-1/6}}(pM)^{-N/6} \right)$$
$$\geq 1 - C_1 \sum_{n=N/L}^{N} \gamma^{(pM)^{n/3}} - C_1 \sum_{n=N}^{\infty} M^{N-n} \gamma^{(pM)^{n/3}} =: p_N$$

Let p_N denote the probability as written above, and $y(x) = \lim_{n\to\infty} y_n(x)$ when it exists. Also for $n \geq LN_0$ let Q_n be the event that $\exists j_n \; x \in K_{j_n}$ $|y_n(x) - y(x)| > \frac{1}{1-(pM)^{-1/6}}(pM)^{-N/6}$. As $\mathbb{P}(Q_n) < M^n(1-p_n)$ and $\sum_{n=LN_0}^{\infty} M^n(1-p_n) \leq C_1 \sum_{n=LN_0}^{\infty} \sum_{m=n/L}^{\infty} M^m \gamma^{(pM)^{m/3}} < \infty$. By Borel-Cantelli Lemma $\mathbb{P}(\limsup_n Q_n) = 0$ which means that Q_n happens for only finitely many n almost surely, which means, that $\exists N_1$ such that the event happens for all M-adic intervals of level greater than N_1 almost surely. □

As $y_N(x)$ is constant on the M-adic intervals of level N for any $x, y \in (lM^{-N}, (l+1)M^{-N})$:

$$|y(x) - y(y)| < 2C_2 b^N$$

3.2.2 The general case

The main part of this section is to prove a similar statement as in Proposition 3.4 with a difference that the statement must hold for all $\alpha \in (0, \pi/2)$ directions (just these, again because of the symmetry), as the statement is similar, we are going to use the same tool-box only with a little modification. The essence of the method is similar to the method seen in the last chapter, in the proof by Simon and Rams, namely we choose finitely many points and finitely many directions, and use that the points in a small enough neighborhood acts similarly. We need this because unlike the other case, here the density functions are not constant on a small interval, also we have a set of directions not just one. Like in chapter two we change the range of the projection to the well-known Δ, this will change the densities, but only with a multiplicative constant.

Proposition 3.6. *There exists $b < 1$ such that almost surely the following holds. For every $\delta > 0$ there exists a $C_3 > 0$ and $N_2 > 0$ such that for all $N > N_2$, for all pairs of points $x, y \in \Delta$, $|x - y| < M^{-N-1}$ and for all $\alpha \in [\delta, \pi/2 - \delta]$ we have*

$$\left| \lim_{m\to\infty} y_m^\alpha(x) - \lim_{m\to\infty} y_m^\alpha(y) \right| < C_3 b^N$$

In particular, the limits exist everywhere.

Before the proof, we need some preparation: first of all $Y(i_{n-1}, j_{n-1}; x, \alpha)$ is Lipschitz function in x and α, as it is the density function of the projection of the Lebesgue measure restricted to squares. Let the Lipschitz constant be in a form of $C_4\delta^{-1}/2$, where C_4 is depending on p, M, δ. As $y_n(x) = p^{-n} \sum_{i_n, j_n \in \mathcal{E}_n - 1} Y(i_{n-1}, j_{n-1}; x, \alpha)$, and $\#\mathcal{E}_n$ is not greater than $2M^n$, we have:

$$|y_n^\alpha(x) - y_n^\alpha(y)| \le 2C_4 p^{-n} M^n \delta^{-1} |x - y|$$

$$|y_n^{\alpha_1}(x) - y_n^{\alpha_2}(x)| \le 2C_4 p^{-n} M^n \delta^{-1} |\alpha_1 - \alpha_2| \tag{3.11}$$

Define two sequence: $\{\alpha_{n,j}\} \subset [\delta, \pi/2 - \delta]$ and $\{x_{n,i}\} \subset \Delta$ both $\delta C_4 p^{5n/6} M^{-7n/6}$-dense, and so can be chosen in a way that both contains at most $C_5\delta^{-1}p^{-5n/6}M^{7n/6}$ elements. Let $T_{n,j} = \{\alpha \in [\delta, \pi/2 - \delta] : \forall l \neq j |\alpha_{n,l-\alpha}| \ge |\alpha_{n,j} - \alpha|\}$, and also $W_{n,i} = \{x \in \Delta : \forall l \neq i |x_{n,l} - x| \ge |x_{n,j} - x|\}$. These sets covers $[\delta, \pi/2 - \delta]$ and Δ respectively, and $\forall \alpha \in T_{n,i}$ $|\alpha - \alpha_{n,i}| \le \delta/2C_4^{-1}p^{5n/6}M^{-7n/6}$ and

$\forall x \in W_{n,j}$ $|x - x_{n,j}| \leq \delta/2C_4^{-1}p^{5n/6}M^{-7n/6}$. Which implies that $\exists\ C_6 > 0$ such that $\forall n > 0$ $\forall x \in W_{i,n}$ and $\alpha \in T_n, j$

$$|y_n^{\alpha}(x)-y_n^{\alpha_{n,,j}}(x_{n,i})| \leq |y_n^{\alpha}(x)-y_n^{\alpha}(x_{n,i})|+|y_n^{\alpha}(x_{n,i})-y_n^{\alpha_{n,,j}}(x_{n,i})| < C_6(pM)^{-n/6} \tag{3.12}$$

Now let $J \subset \Delta$ an interval of length M^{-N}, if we choose n such that

$$N > \frac{7n}{6} + \frac{5n}{6}\log_M \frac{1}{p} + \log_M \frac{C_4}{\delta}$$

then the variation of y_n^{α} in I is – by the Lipschitz property – bounded above by the Lipschitz constant times the length of the interval, in this case $M^{-N}C_4p^{-n}M^n\delta^{-1}$ which by the definition of n is less than $(pM)^{-n/6}$, hence for each α the variation of y_n^{α} inside J is not greater than $(pM)^{-n/6}$. Thus the next proposition holds:

Proposition 3.7. *There exist $L', L'' > 0$ such that for any N if $J \subset \Delta$ is an interval of length M^{-N} and $n \leq L'N - L''$ then for each α the variation of y_n^{α} in J is not greater than $(pM)^{-n/6}$.*

The next proposition is similar to Proposition 3.3, but we have to change the value of N_0, namely let N_0 be the smallest number for which

$$1 + (pM)^{-N_0/3} + 2C_6(pM)^{-N_0/6} < (pM)^{1/8}$$

Proposition 3.8. *If for $n > N_0$, j and $\forall x \in J$: $y_n^{\alpha_{n,j}}(x) < (pM)^{n/3}$, then*

$$\mathbb{P}(\forall m \geq n\ \forall x \in J\ \forall \alpha \in T_{n,j}\ |y_{m+1}^{\alpha}(x) - y_m^{\alpha}(x)| < (2C_6+1)(pM)^{-n/6})$$
$$> 1 - C_1C_5^2\delta^{-2}p^{-5n/3}M^{7n/3}\sum_{m=n}^{\infty}\gamma^{(pM)^{m/3}}$$

Proof. First what we need to do is make a mesh in $J \times T_{n,j}$ with the same properties as above. In that way at every level we work with a finer and finer mesh, and also we know what happens at the grid points, and we can approximate what happens in between them. The first level mesh is $J_{n+1,i} \times T_{n+1,j}$.

$$\left|y^{\alpha}_{m+1}(x) - y^{\alpha}_{m}(x)\right| \leq \left|y^{\alpha}_{m+1}(x) - y^{\alpha_{m,j}}_{m+1}(x_{m,i})\right| + \left|y^{\alpha_{m,j}}_{m+1}(x_{m,i}) - y^{\alpha_{m,j}}_{m}(x_{m,i})\right| + \left|y^{\alpha_{m,j}}_{m}(x_{m,i}) - y^{\alpha}_{m}(x)\right|$$

hence for n+1 using equation 3.12 and Proposition 3.2:

$$\begin{aligned}
&\mathbb{P}(\exists x \in J\ \exists \alpha \in T_{n,j}\ \left|y^{\alpha}_{n+1}(x) - y^{\alpha}_{n}(x)\right| > (2C_6+1)(pM)^{-n/6}) \\
&\leq \mathbb{P}(\exists i\ \exists x \in J_{n+1,i}\ \exists k\ \exists \alpha \in T_{n,j} \cap T_{n+1,k}\ \left|y^{\alpha}_{n+1}(x) - y^{\alpha_{(n+1),j}}_{n+1}(x_{(n+1),i})\right| \\
&+ \left|y^{\alpha_{n+1,k}}_{n+1}(x_{n+1,i}) - y^{\alpha_{n+1,k}}_{n}(x_{n+1,i})\right| + \left|y^{\alpha_{n+1,k}}_{n}(x_{n,i}) - y^{\alpha}_{n}(x)\right| \\
&> (2C_6+1)(pM)^{-n/6}) \\
&\leq \mathbb{P}(\exists x_{n+1,i} \in J_{n+1,i} \exists \alpha \in T_{n+1,k} \left|y^{\alpha_{n+1,j}}_{n+1}(x_{n+1,i}) - y^{\alpha_{n+1,k}}_{n}(x_{n,i})\right| > \\
&(pM)^{-n/6}) < C_1 C_5^2 \delta^{-2} p^{-5n/3} k^{7n/3} \gamma^{(pM)^{n/3}}
\end{aligned}$$

thus, using the method as in the proof of Proposition 3.3 the assumption can be proven. □

Let $N > (LN_0 + L'')/L'$ and let $n = \lfloor L'N - L'' \rfloor$. We can choose J_i intervals with length M^{-N} in Δ such that $\forall x, y, \in \Delta$ if $|x-y| \leq M^{-N-1}$ then $\exists i$:$x, y \in J_i$, the number of these intervals is less than $4M^N$. Now we can use Proposition 3.3 and Proposition 3.7 to give a lower bound for the probability that for all $x \in J_i$ and all $\alpha_{n,j}$ $y^{\alpha_{n,j}}_{n}(x)$ is smaller than $(pM)^{n/4}$ and its variation in J_i is not greater than $(pM)^{-n/6}$. Namely the lower bound is

$$p'_N = 1 - C_1 C_5 \delta^{-1} p^{-5n/6} M^{7n/6} \sum_{m=n/L}^{n} \gamma^{(pM)^{m/3}} \tag{3.13}$$

This needs a little explanation because Proposition 3.3 was stated for the vertical case. What we do is estimate the probability $y_n^{\alpha_{n,j}} > (pM)^{n/4}$ for a given $\alpha_{n,j}$ and then as $\#\{\alpha_{n,j}\} \leq C_5 \delta^{-1} p^{-5n/6} M^{7n/6}$ the probability that the complement event happens for all $\alpha_{n,j}$ will be the one in equation 3.13. Now we can apply Proposition 3.8 and similarly to the vertical case prove that with probability

$$p_N > 1 - C_1 C_5 \delta^{-1} p^{-5n/6} M^{7n/6} \sum_{m=n/L}^{n} \gamma^{(pM)^{m/3}} - C_1 C_5^2 \delta^{-2} p^{-5n/3} M^{7n/3} \sum_{m=n}^{\infty} \gamma^{(pM)^{m/3}}$$

for all $\alpha \in [\delta, \pi/2 - \delta]$ and $x, y \in I_i$ $\lim_{m\to\infty} y_m^\alpha(x)$ exists and

$$\left| \lim_{m\to\infty} y_m^\alpha(x) - \lim_{m\to\infty} y_m^\alpha(y) \right| < (pM)^{-n/6} + \sum_{m=n}^{\infty} (2C_6 + 1)(pM)^{-m/6} = \left(1 + \frac{2C_6 + 1}{1 - (pM)^{-1/6}}\right)(pM)^{-n/6} \tag{3.14}$$

And as we have $4M^N$ intervals the probability that the event does not happen for at least one interval is less than $4M^N(1 - p_N)$, summing this up, and using Borel-Cantelli Lemma we have that for every sufficiently large N_2 the assertion in Proposition 3.6 holds. As δ is arbitrarily close to 0 Theorem 3.1 follows if $\delta \in (0, \pi/2)$, and in the horizontal and vertical case the statement of the theorem was proved in the previous section.

Chapter 4

Conclusions

The main part of this thesis was the proof of two statements about the projection of the Mandelbrot percolation fractal (see Chapter 2), and the projection of the natural measure in the homogeneous case (see Chapter 3). The methods in the proofs give us a useful toolbox for considering the properties of the inhomogeneous Mandelbrot percolation fractal. As we mentioned before we know a lot about the homogeneous case, but very little about the inhomogeneous one. A possible way to move on is to consider the absolute continuity of the projection of the natural measure with respect to the Lebesgue measure in the later case.

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