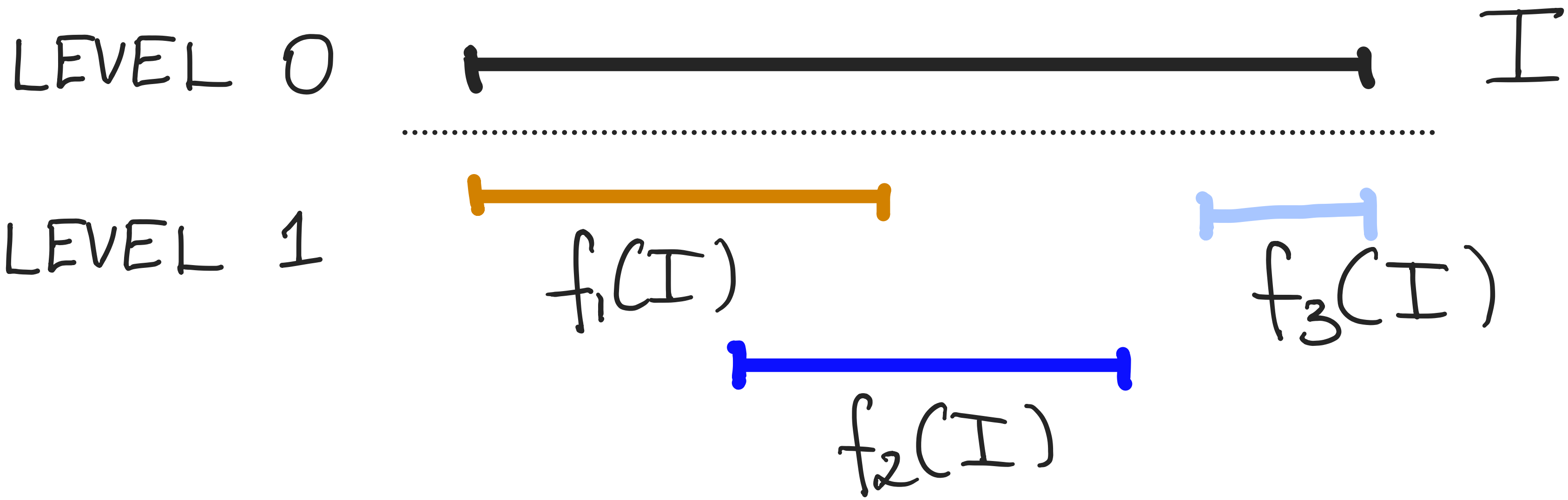
RANDOM SELF-SIMILAR SETS ON THE LINE

VILMA ORGOVÁNYI
JOINT WITH KÁROLY SIMON

16/05/23

COIN TOSSING SELF SIMILAR SETS ON THE LINE

- $\mathcal{F} = \{ f_i(x) = r_i x + t_i \}_{i=1}^{M}$, $r_i \in (0,1)$, $t_i \in \mathbb{R}$.
- CHOOSE $p \in [0,1]$ AND A COIN $\mathbb{P}(\text{H}) = p$, $\mathbb{P}(\text{T}) = 1-p$



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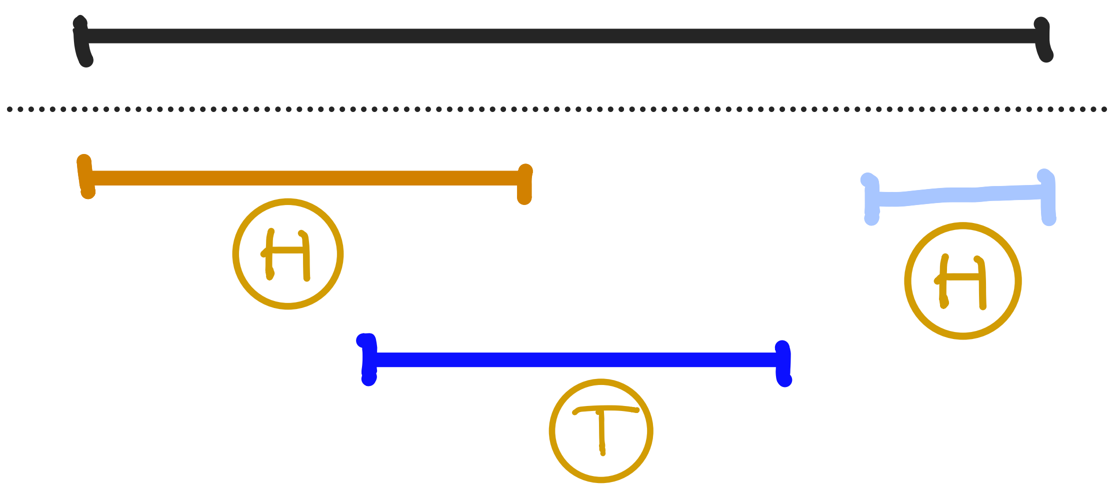
LEVEL 0

LEVEL 1



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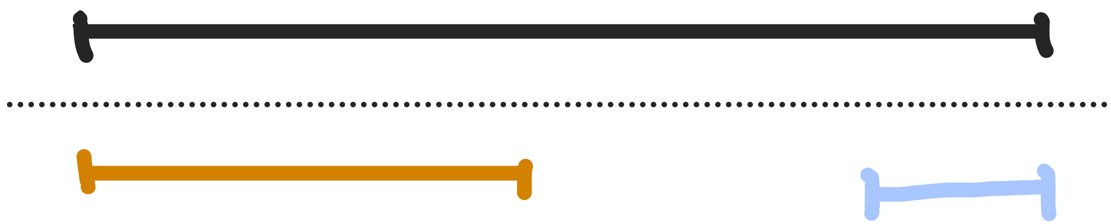

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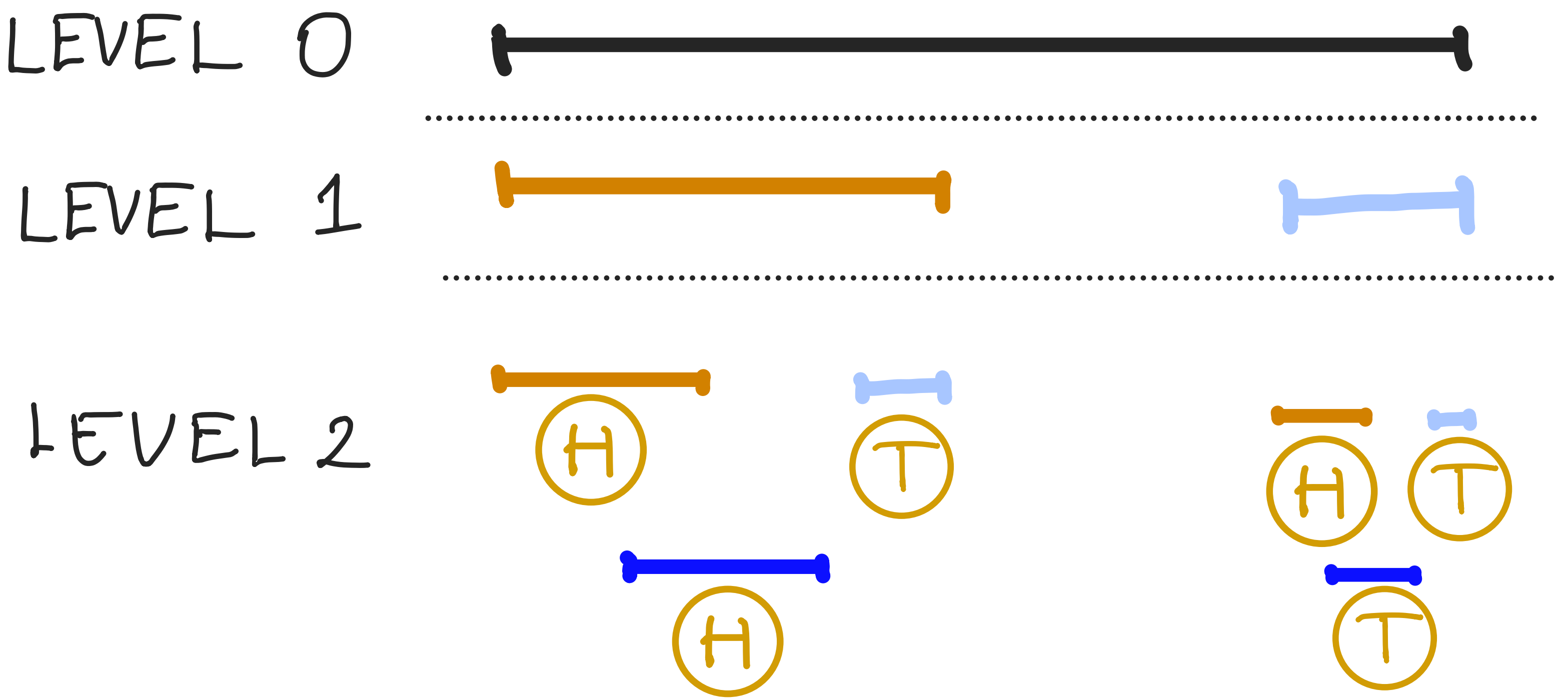

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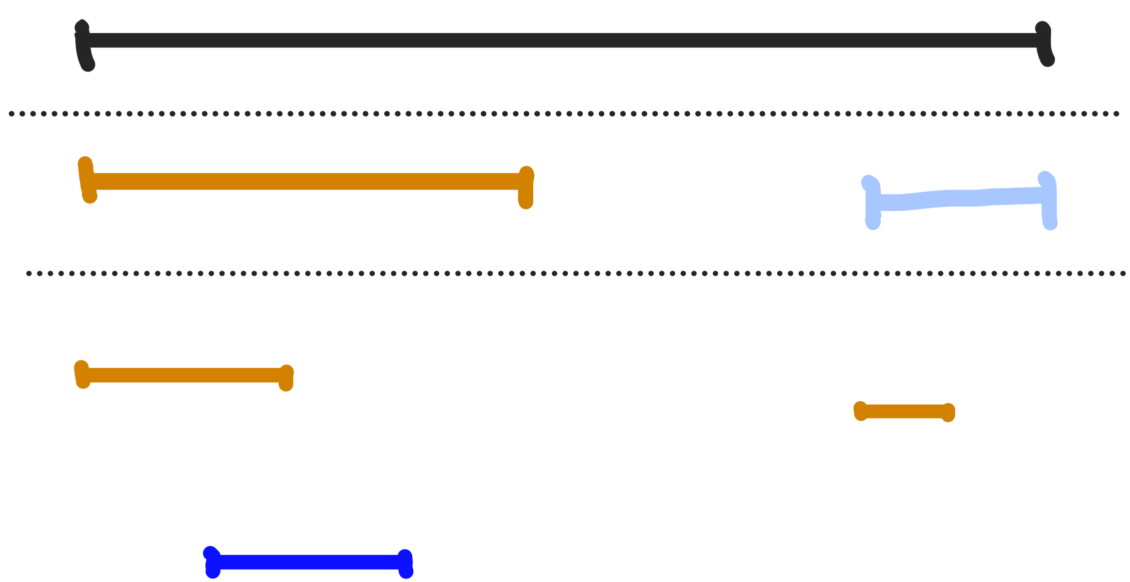
LEVEL 2



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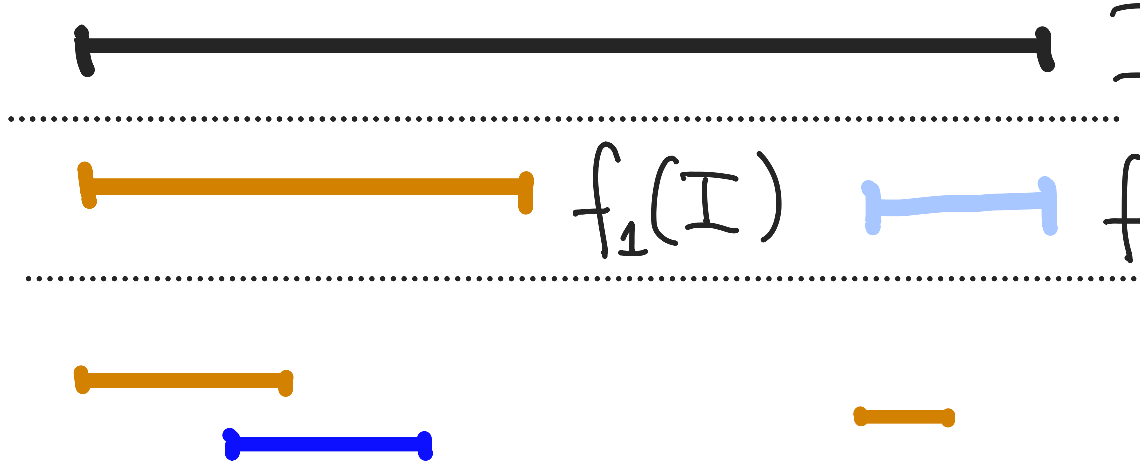

LEVEL 0 $\qquad I$

LEVEL 1 $\qquad f_1(I) \qquad f_3(I)$

LEVEL 2 $\qquad f_1 f_2(I) \; \dots \; f_3 f_1(I)$

$$\Lambda(\mathcal{F}, p) = \bigcap_{n} \bigcup_{(i_1 \dots i_n) \in \mathcal{E}_n} f_{i_1} f_{i_2} \dots f_{i_n}(I)$$

INTEGER COIN TOSSING SELF-SIMILAR IFSs ON THE LINE

- $\mathcal{F} = \{ f_i(x) = \frac{1}{L} x + t_i \}_{i=1}^{M}$, $2 \leq L \in \mathbb{N}$, $t_i \in \mathbb{Q}$.

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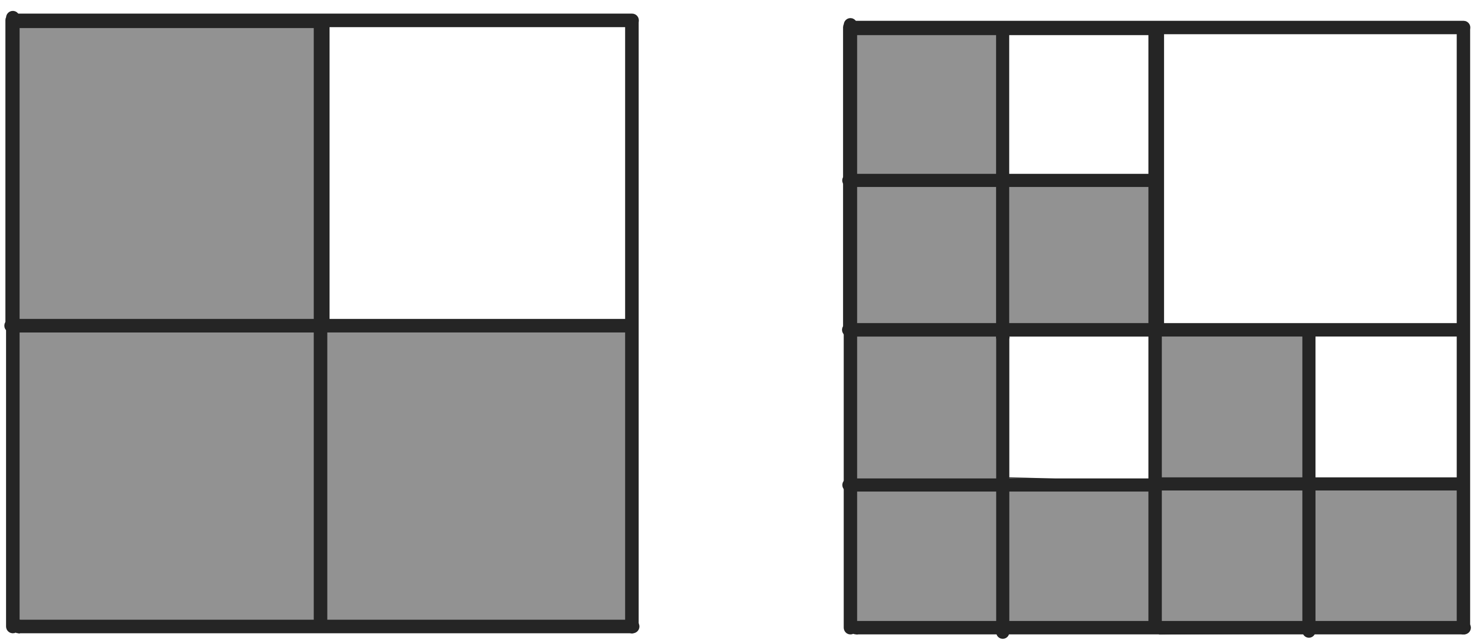
E.g. RATIONAL PROJECTIONS OF THE RIGHT ANGLED SIERPIŃSKI CARPET



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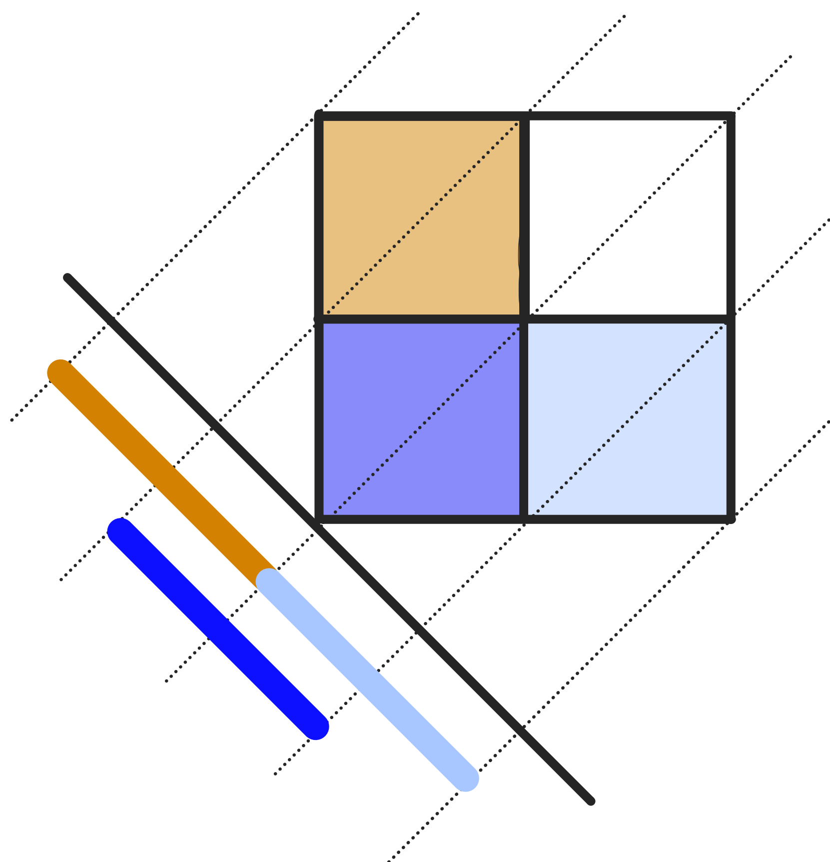


QUESTIONS – SIZE (DEP. ON p)?

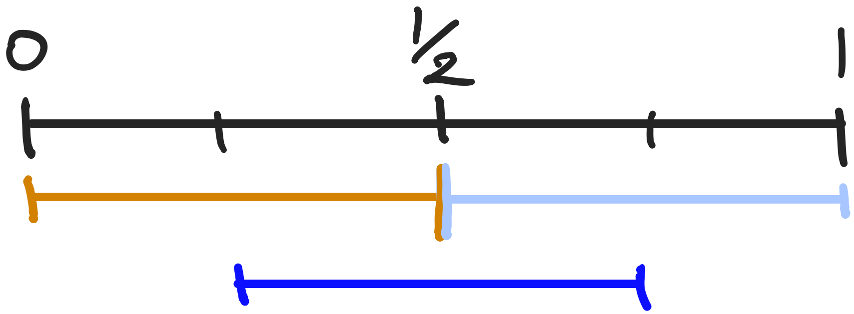
(1) DIMENSION OF THE SET?

(2) POSITIVITY OF LEBESGUE MEASURE?

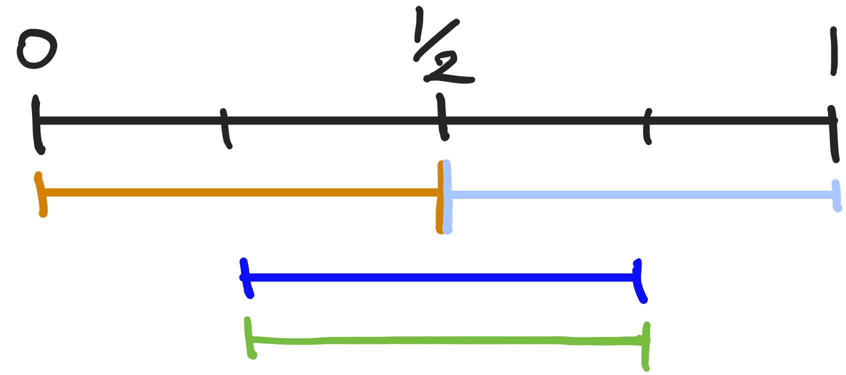
(3) EXISTENCE OF INTERIOR POINTS?

EXAMPLES

(1)



(2)



EXAMPLES

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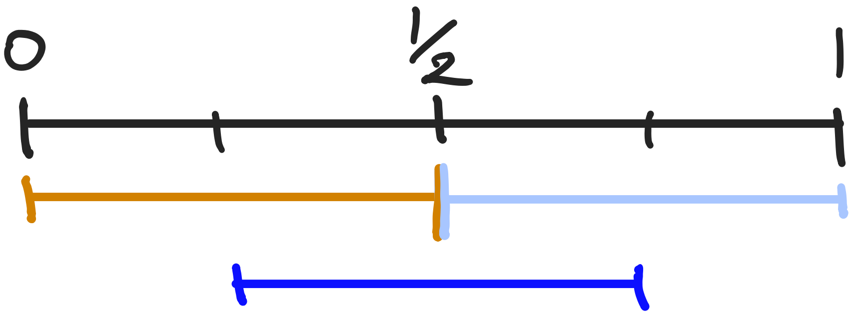

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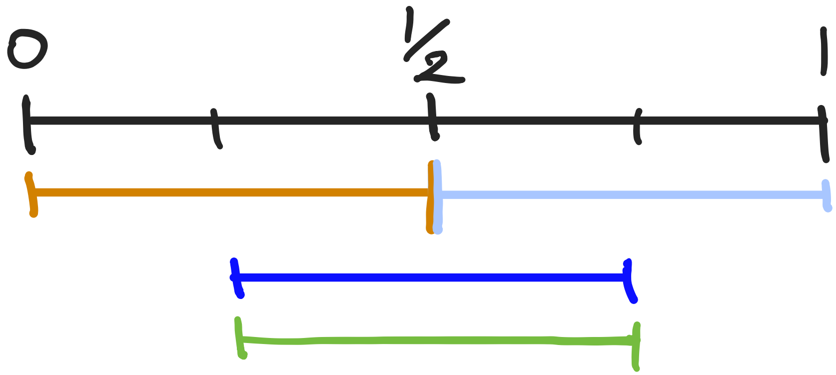
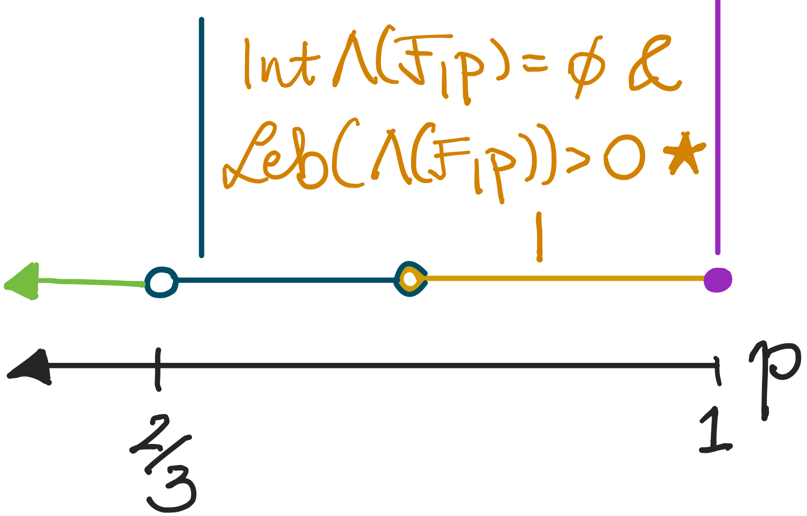


$\dim_S \Lambda(\mathcal{F},p) = \dfrac{\log 3p}{\log 2}$

$\dim_S \Lambda(\mathcal{F},p) > 1$

$\dim_H \Lambda(\mathcal{F},p) < 1$

$\mathrm{Int}\,\Lambda(\mathcal{F},p) \neq \emptyset$ ★



★ A.S. COND. ON NON-EXTINCTION

EXAMPLES

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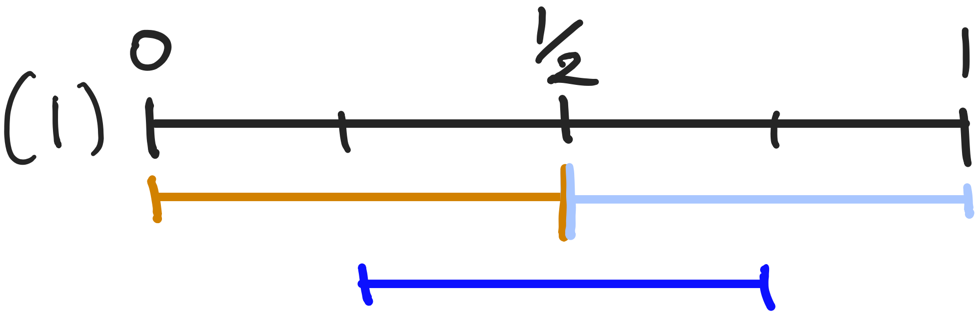

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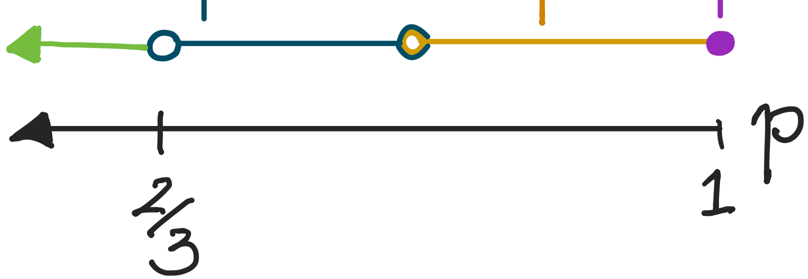
$\mathrm{Int}\,\Lambda(\mathcal{F},p) = \emptyset$ &

$\mathcal{L}eb(\Lambda(\mathcal{F},p)) > 0$ ★



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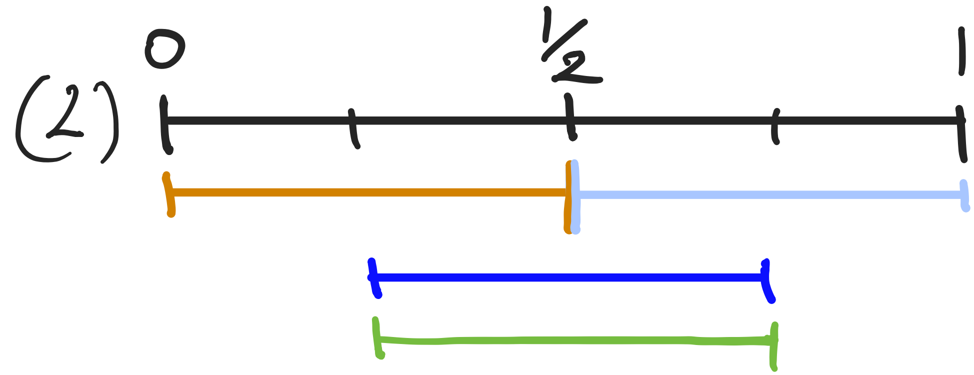
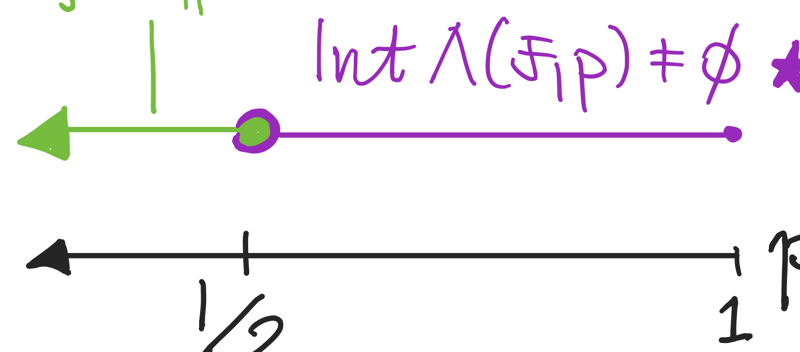


$$\dim_S \Lambda(\mathcal{F},p) = \frac{\log 4p}{\log 2}$$

(Follows from the work of Simon & Rams)

$\dim_S \Lambda(\mathcal{F},p) \leq 1$

$\mathrm{Int}\,\Lambda(\mathcal{F},p) \neq \emptyset$ ★



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EXPECTATION MATRICES

- THE POSITIVITY OF LEB. MEASURE & EXISTENCE OF INTERIOR POINTS CODED INTO EXPECTATION MATRICES.

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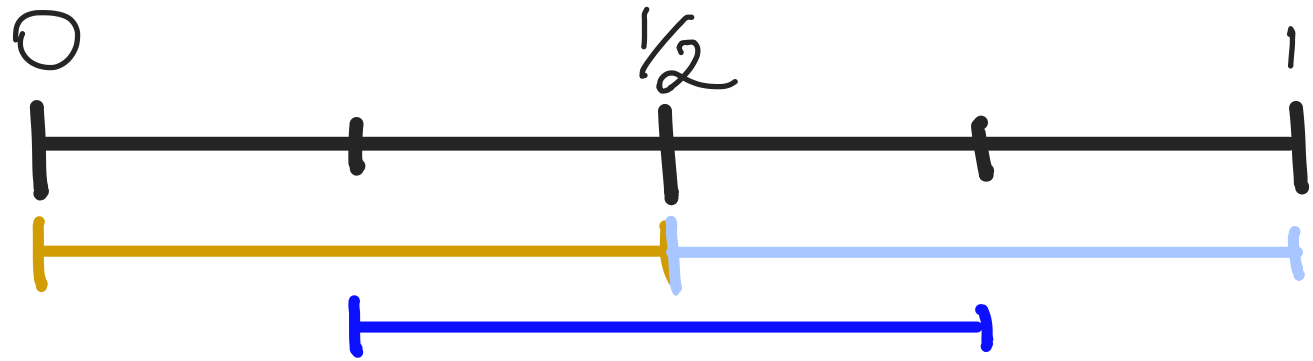
- EXPECTATION MATRICES FOR THE RANDOM SYSTEM = $p \cdot$ MATRICES CORR. TO THE DETERMINISTIC SYSTEM.

$$\Downarrow$$

WE CONSIDER THE DETERMINISTIC SYSTEM NOW.

MATRICES CORRESPONDING THE
DETERMINISTIC IFS (HEURISTIC)



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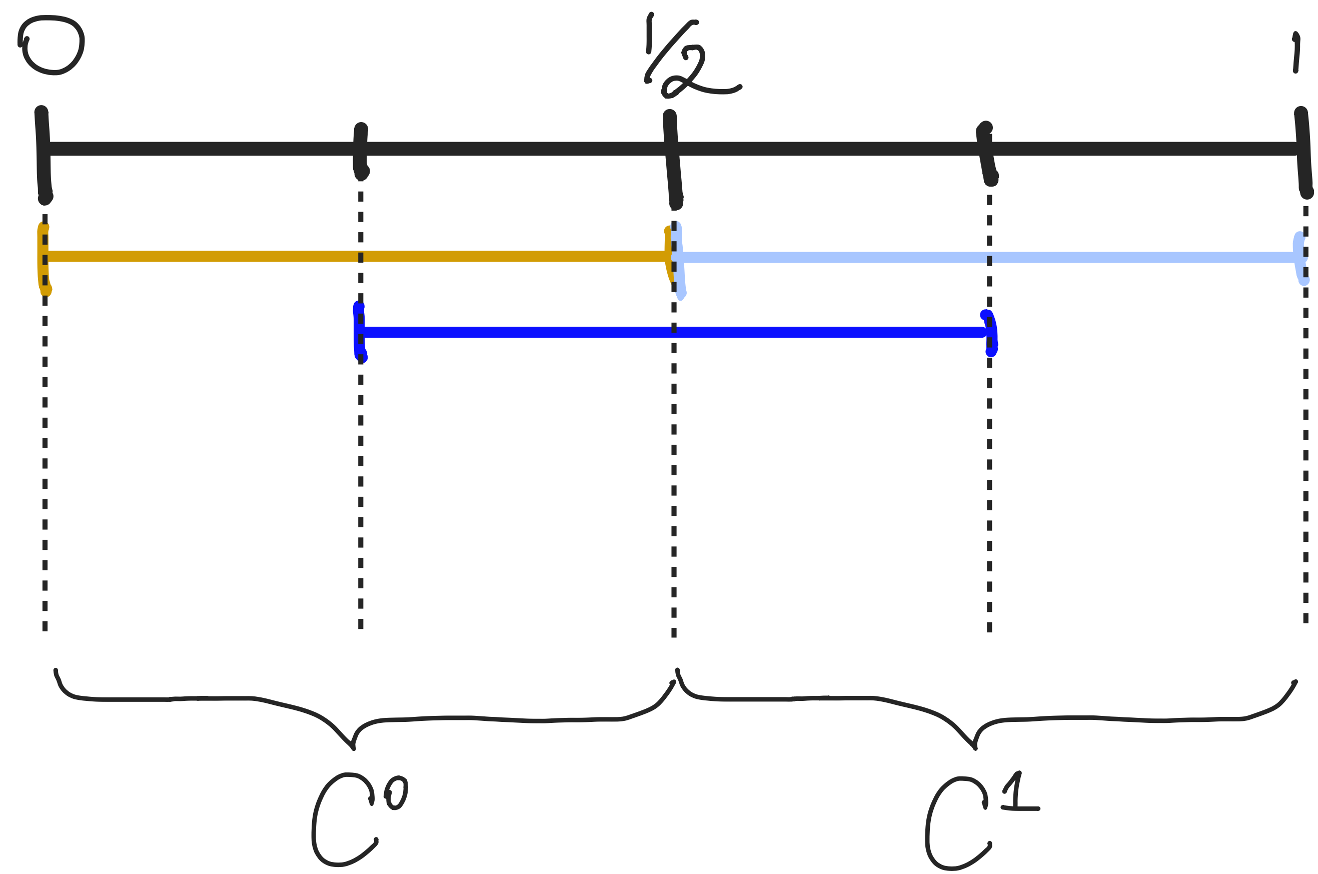

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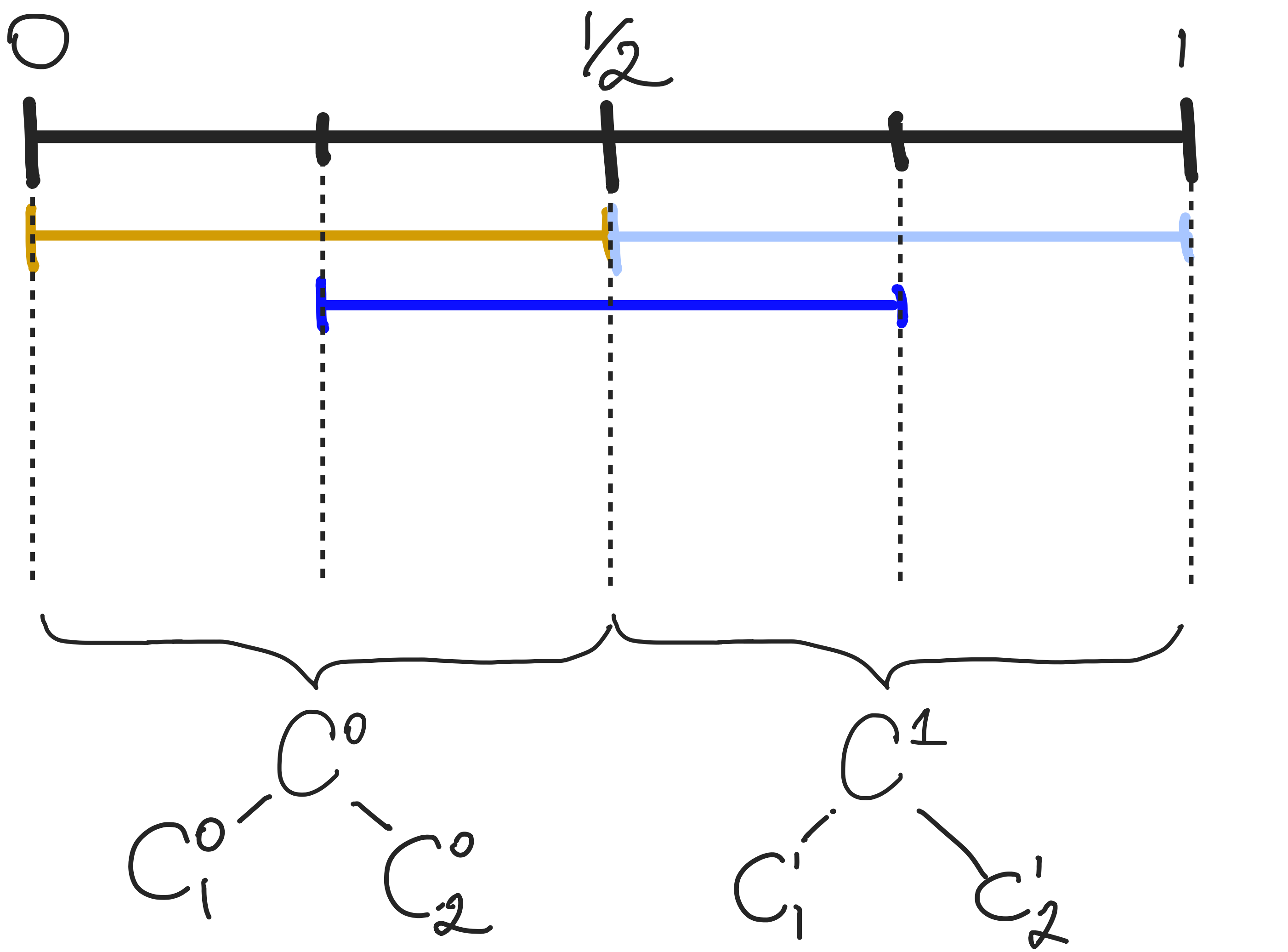

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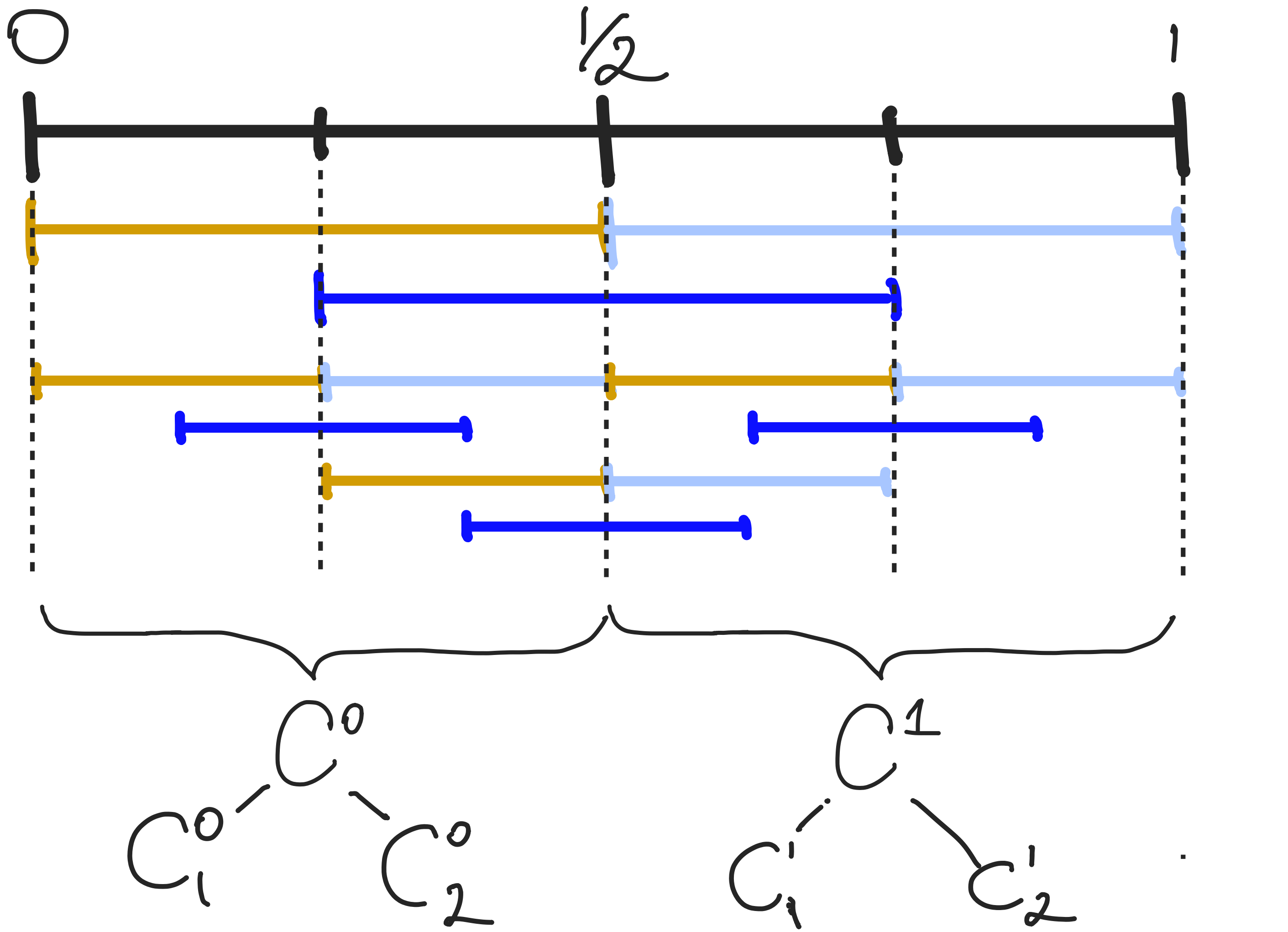

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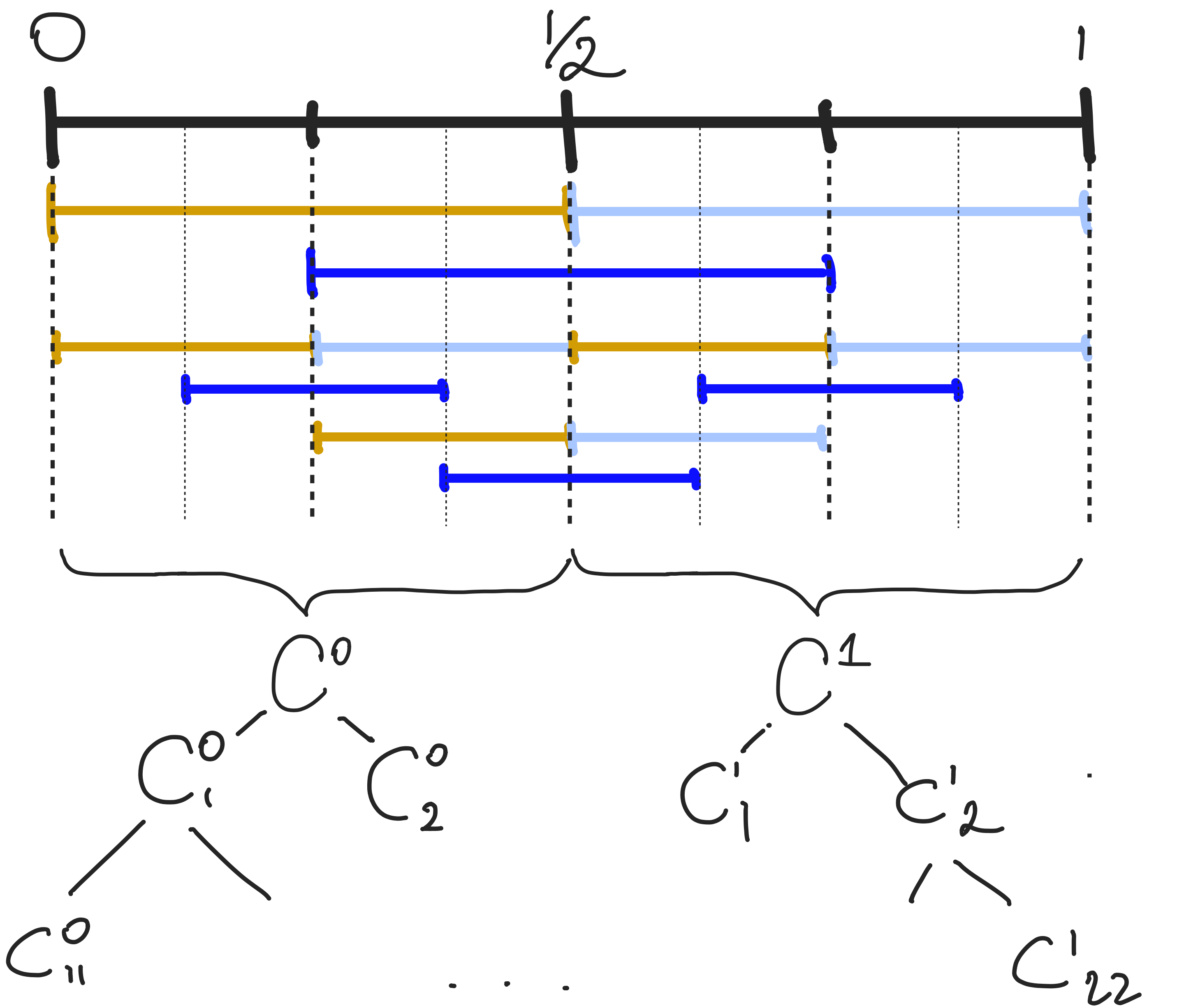


MATRICES CORRESPONDING THE DETERMINISTIC IFS (HEURISTIC)

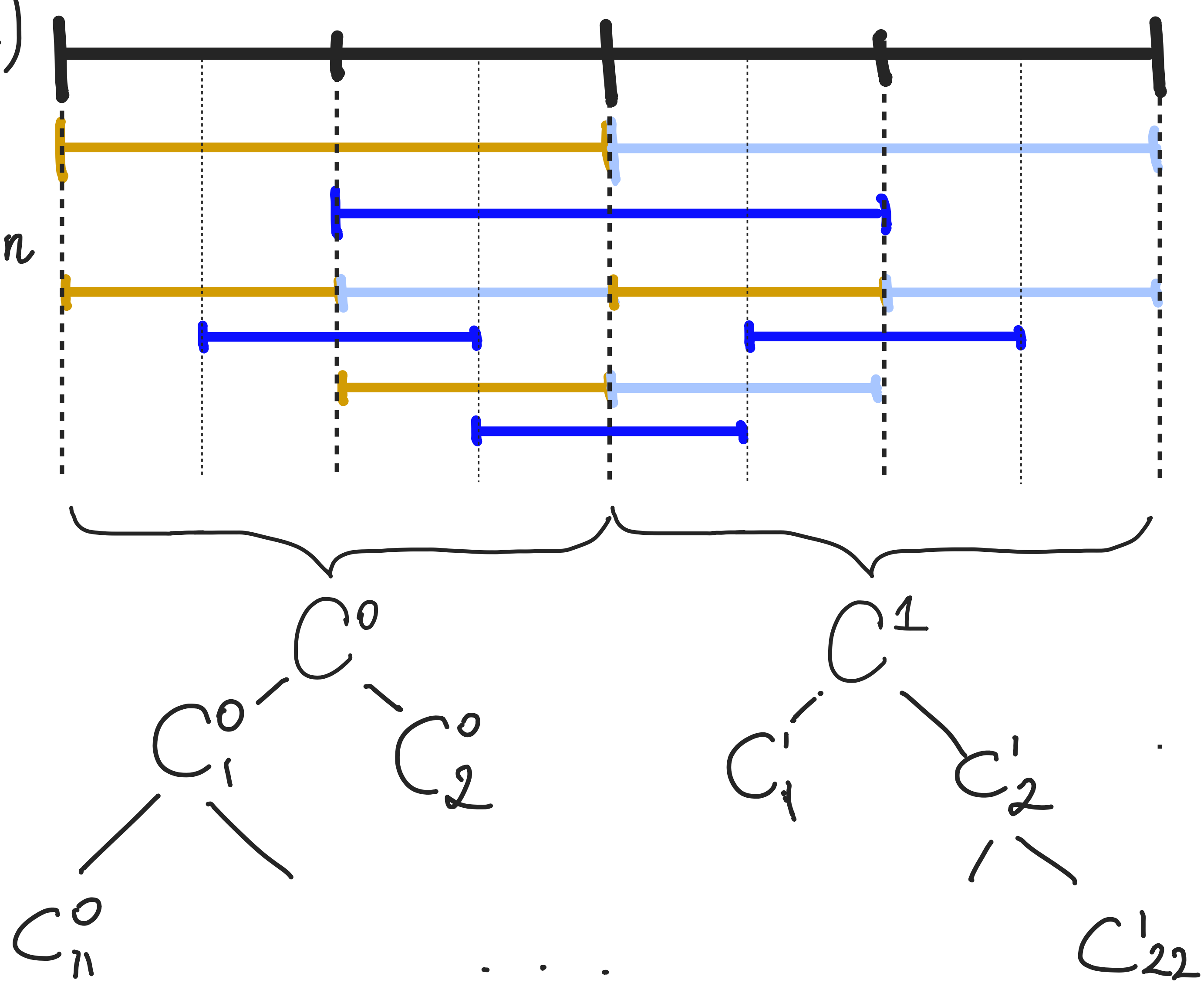
A_1, A_2 (2×2) matrices.

For $(i_1, \ldots, i_n) \in \{0,1\}^n$

$A_{i_1} \ldots A_{i_n}$ describes

$C^j_{i_1 \ldots i_n}$

$j \in \{0,1\}$



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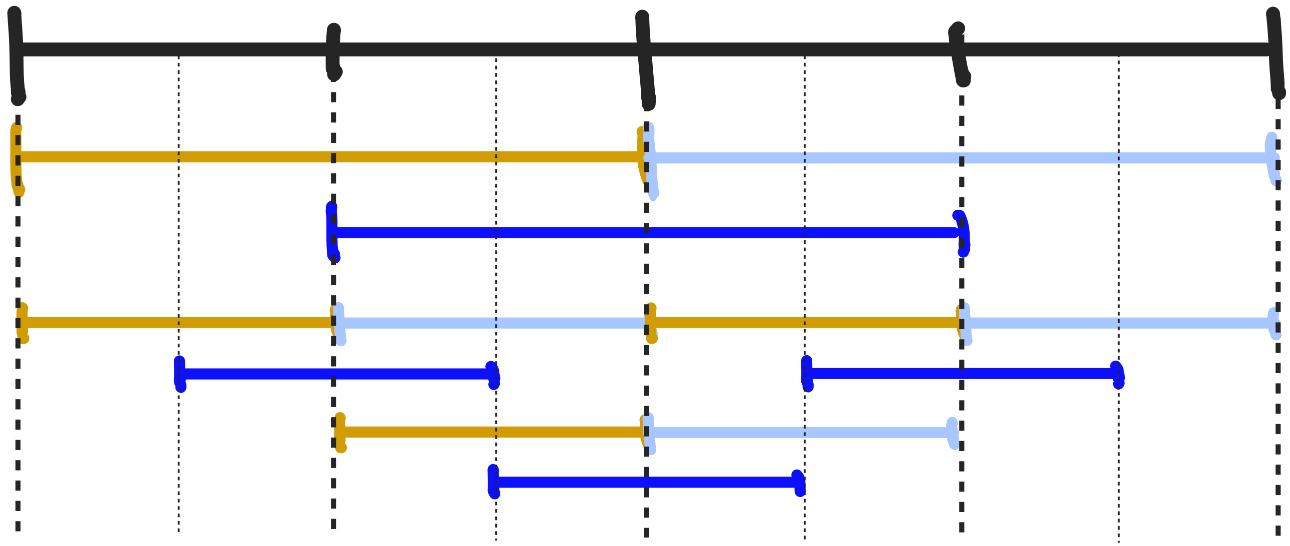

REMARK: RATIONAL PROJECTION/ INTEGER IFS CONDITION IS IMPORTAN

PRESSURE FUNCTION

$\mathcal{A} = \{A_1, \dots, A_\ell\}$; $\|A\| = \sum_i \sum_j |A_{ij}|$; $[\ell] = \{1, \dots, \ell\}$

$$\boxed{P(t) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{(i_1 \dots i_n) \in [\ell]^n} \|A_{i_1 \dots i_n}\|^t}$$

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- $\exists r : \left(\sum_{i=1}^{\ell} A_i\right)^r$ IS STRICTLY POSITIVE

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FENG-LAU (2002), FENG

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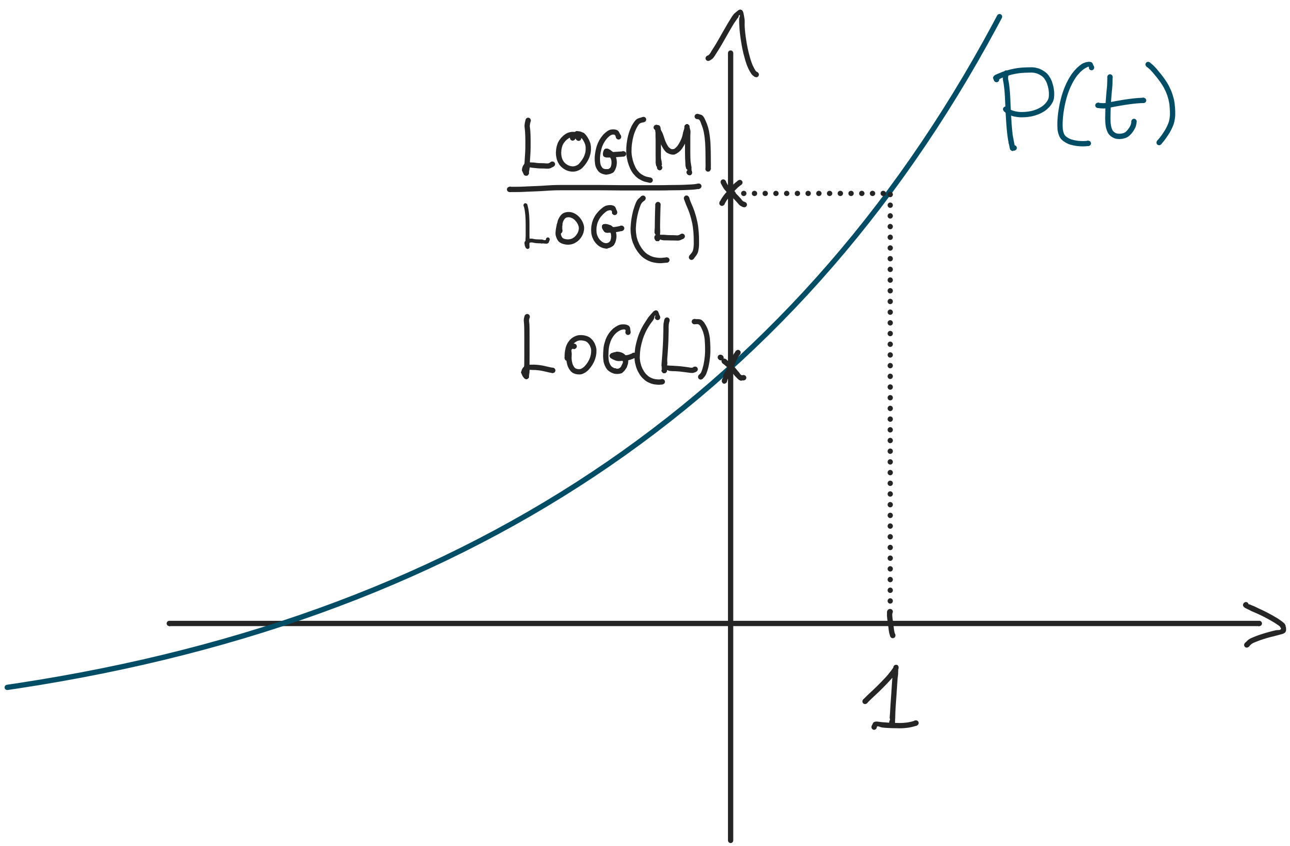
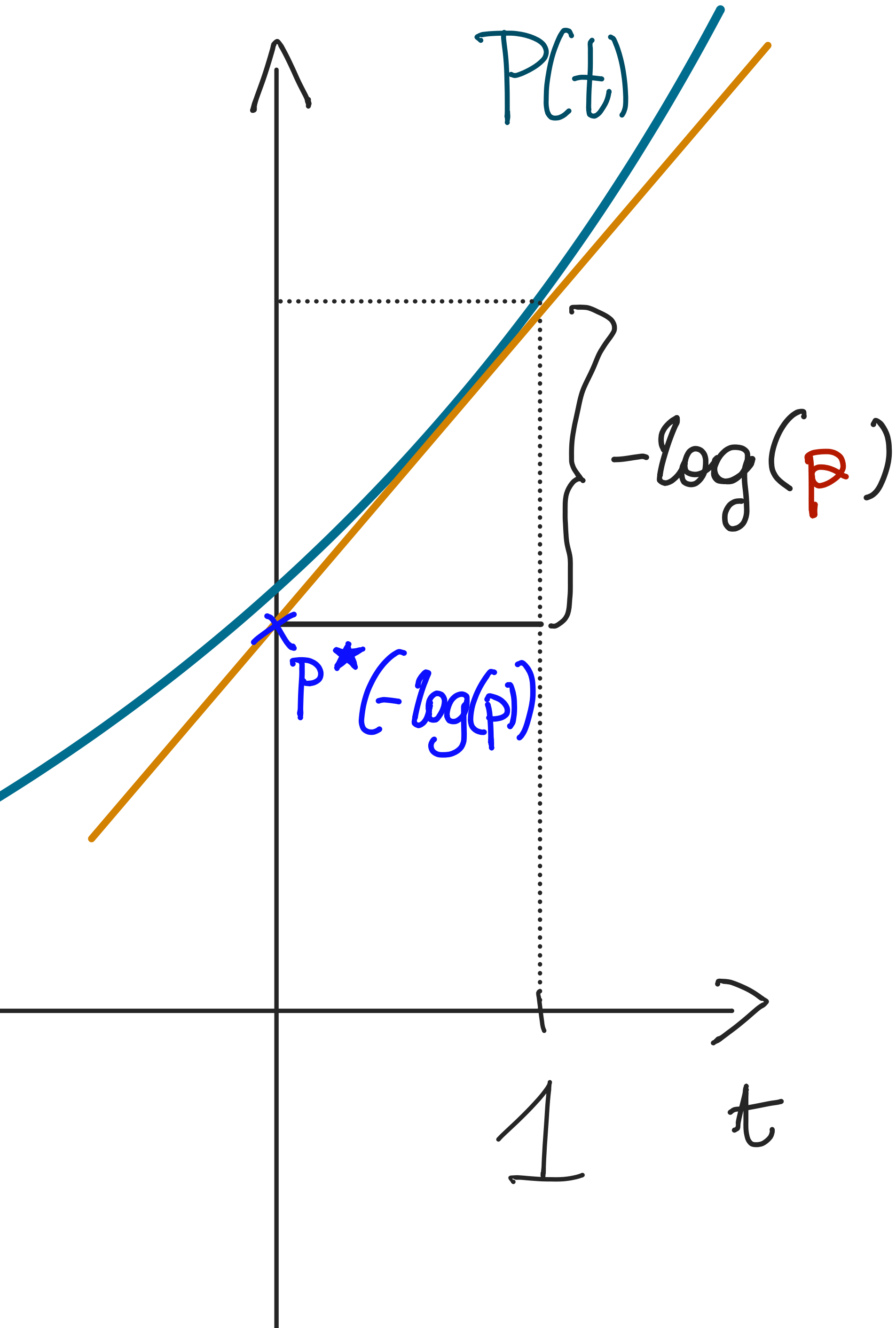


(A.S.) DIMENSION OF THE SET



$$P(t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{(i_1\dots i_n)\in[\ell]^n} \|A_{i_1\dots i_n}\|^t$$

$$\mathrm{DIM}\,\Lambda(F, p) \leq \inf_{t>0} \{ t \cdot \log(p) + P(t) \} = P^*(-\log(p)) \quad \text{A.S.}$$

POSITIVITY OF LEBESGUE MEASURE & NON-EXISTENCE OF INTERIOR POINTS

- FURTHER ASSUMPTIONS ON THE MATRICES $\mathcal{A} = \{A_1, \dots, A_\ell\}$

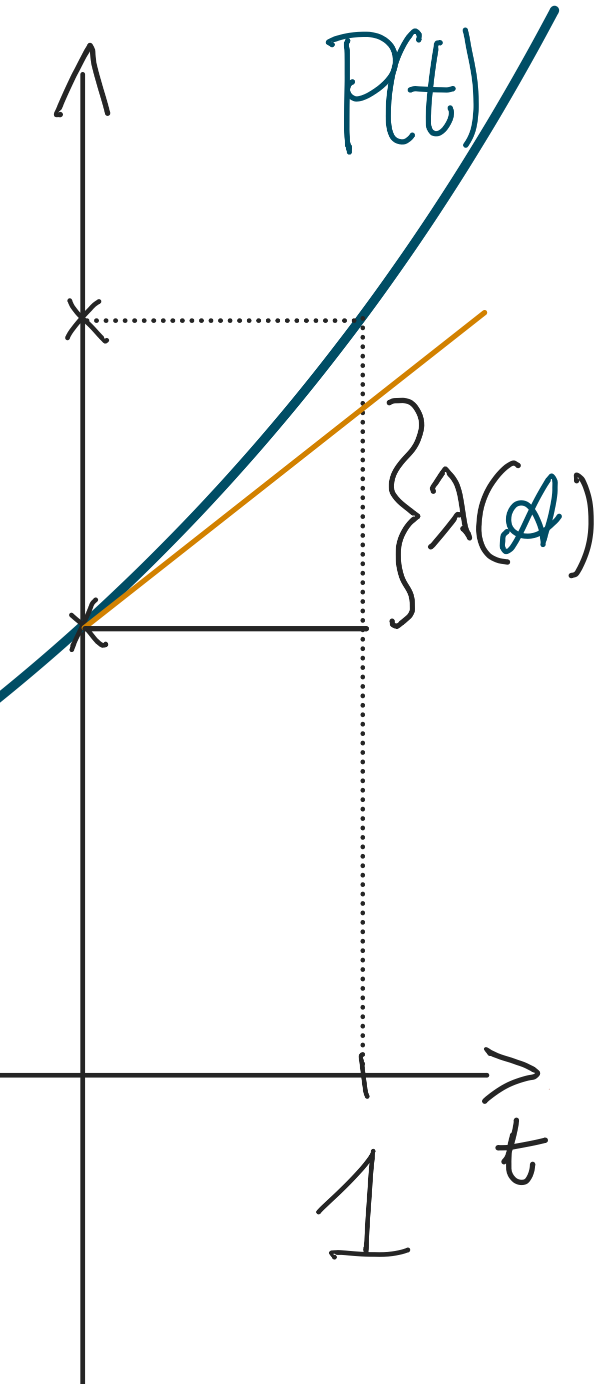
(1) $\exists\, i_1, \dots, i_n \in [\ell]^n$ $A_{i_1} \cdot \ldots \cdot A_{i_n}$ IS STRICTLY POSITIVE

(2) A_i IS ALLOWABLE $\forall i \in [\ell]$

$\Updownarrow$

IN EVERY ROW & COLUMN IT CONTAINS A POSITIVE ELEMENT

Positivity of Lebesgue Measure



Lyapunov-Exponent:

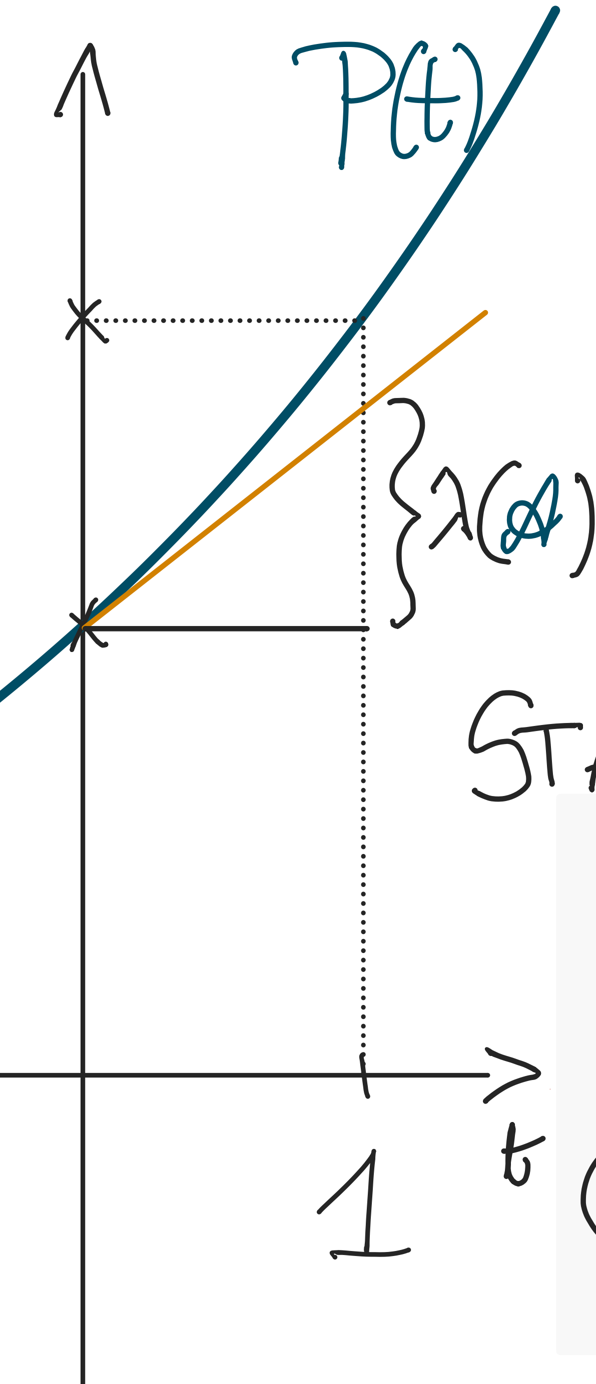
$\mu = (1/k, \dots, 1/k)^{\mathbb{N}}$, for μ a.e. $\underline{i}$

$$\lambda(\mathcal{A}) = \lambda(\mathcal{A}, \underline{i}) = \lim_{n\to\infty} \frac{1}{n} \log\left(\|A_{i_1 \dots i_n}\|\right)$$

BÁRÁNY, RAMS ||

$$\lim_{t\to 0+} P'(t)$$

POSITIVITY OF LEBESGUE MEASURE



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BÁRÁNY, RAMS ||

$$\lim_{t \to 0+} P'(t)$$

STATEMENT (K.S., V.O.):

(A) $\log(p) + \lambda(\mathcal{A}) > 0 \Rightarrow \mathrm{Leb}(\Lambda(\mathcal{F}, p)) > 0$ A.S. ON NON-EXTINCTION

(B) $\log(p) + \lambda(\mathcal{A}) < 0 \Rightarrow \mathrm{Leb}(\Lambda(\mathcal{F}, p)) = 0$ A.S.

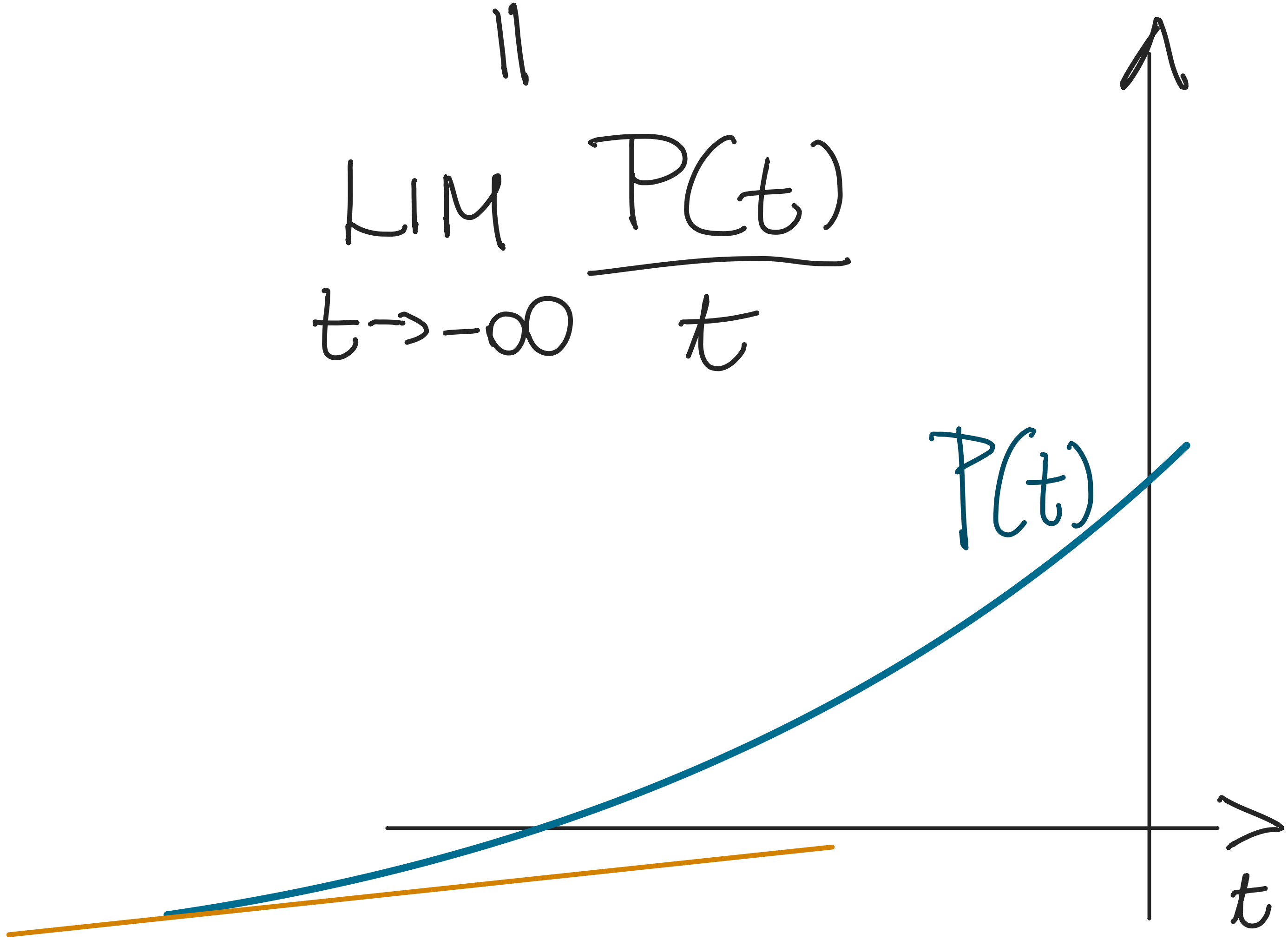
NON-EXISTENCE OF INTERIOR POINTS

LOWER SPECTRAL RADIUS

$$\underline{\rho}(\mathcal{A}) = \lim_{n\to\infty} \min\left\{ \tfrac{1}{n}\log\|A_{i_1}\cdots A_{i_n}\|,\ i_1\dots i_n \in [k]^n \right\}$$

$$\| $$

$$\lim_{t\to-\infty} \frac{P(t)}{t}$$



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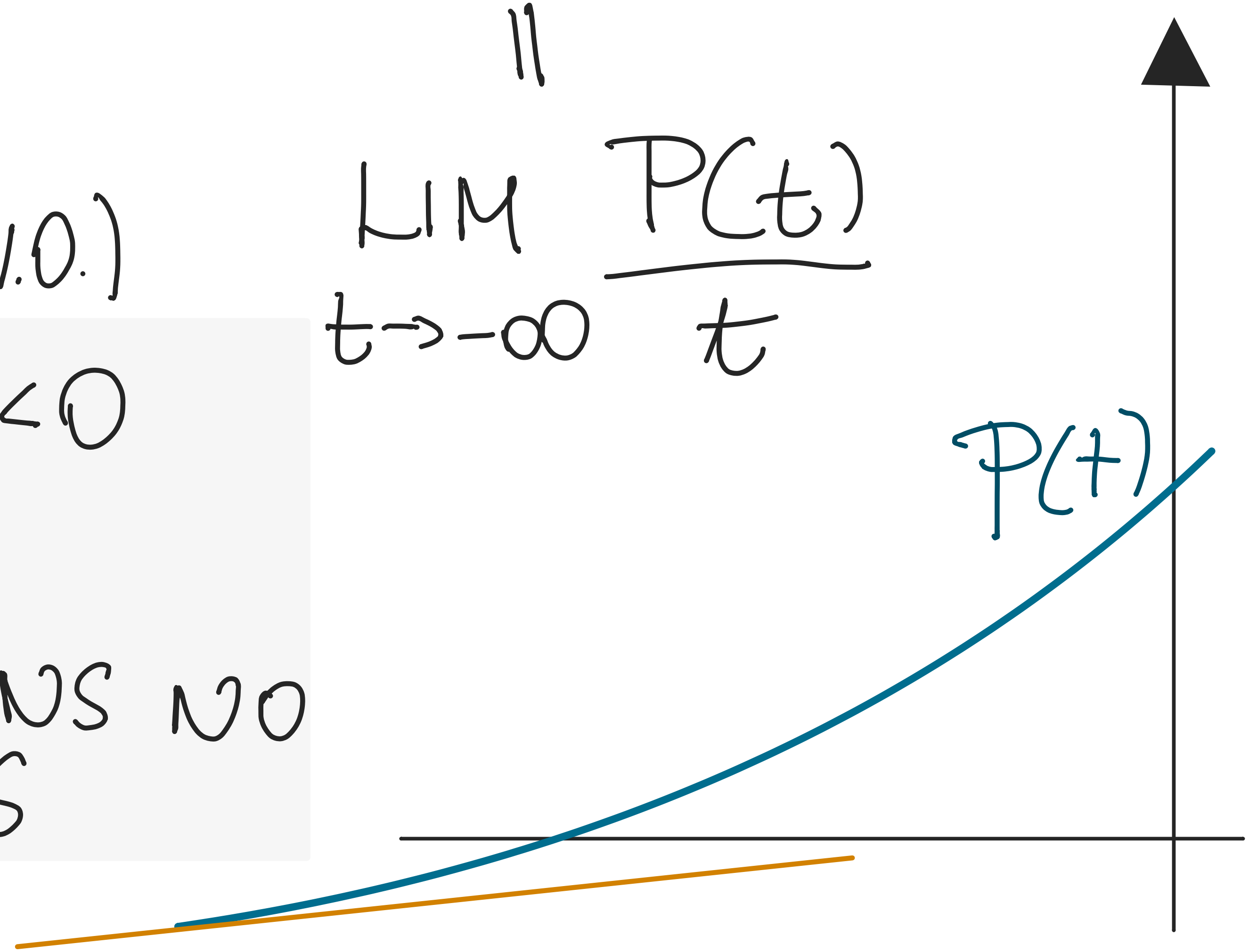
STATEMENT (K.S., V.O.)

$$\log(p) + \underline{\varsigma}(\mathcal{A}) < 0$$

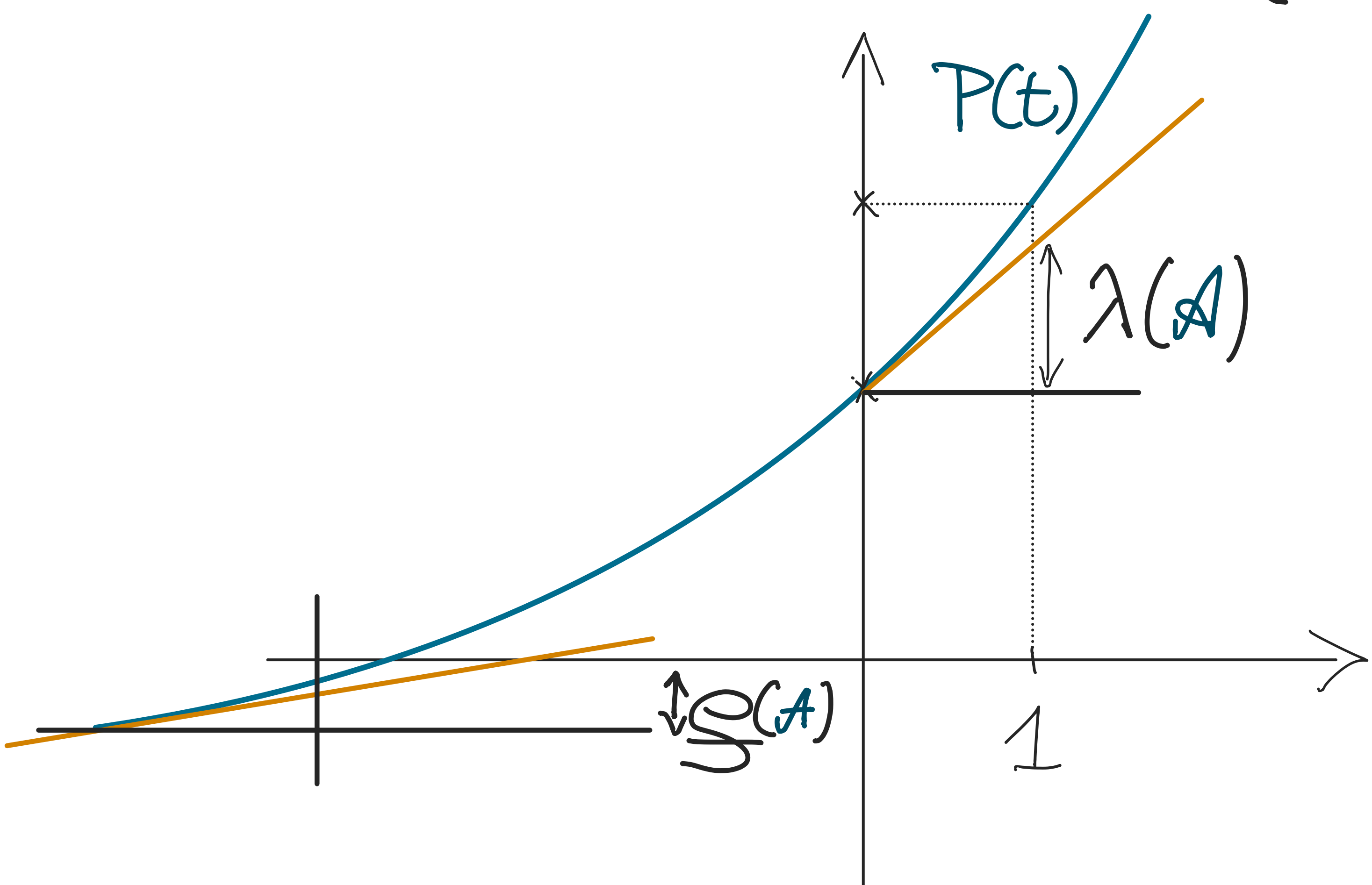
$$\Downarrow$$

$\Lambda(\mathcal{F}, p)$ CONTAINS NO INTERVALS A.S



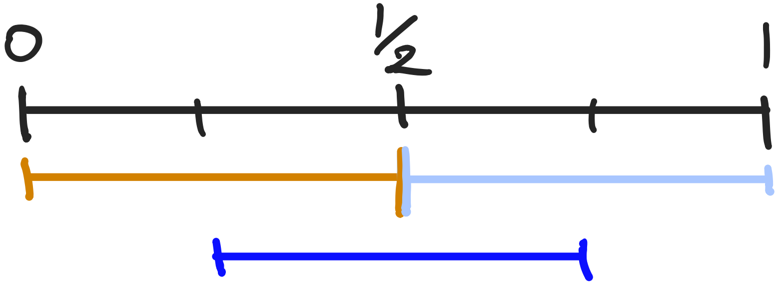
CONCLUSION & FURTHER QUESTIONS

- Q: UNDER WHAT CONDITIONS CAN WE GUARANTEE THAT $\mathrm{Leb}(\Lambda(\mathcal{F}, p)) > 0$ A.S. ON NON-EXTINCTION BUT $\mathrm{INT}(\Lambda(\mathcal{F}, p)) = \emptyset$ A.S.?

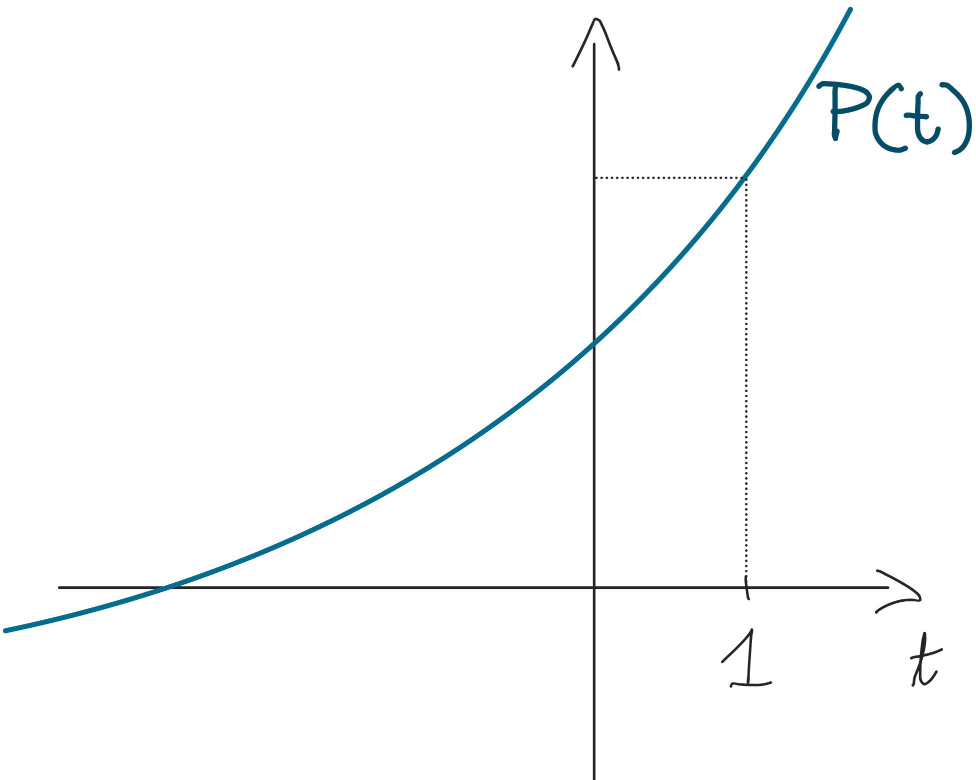


EXAMPLES REVISITED

(1)



$$0 = \mathcal{S}(\mathcal{A}) < \lambda(\mathcal{A})$$



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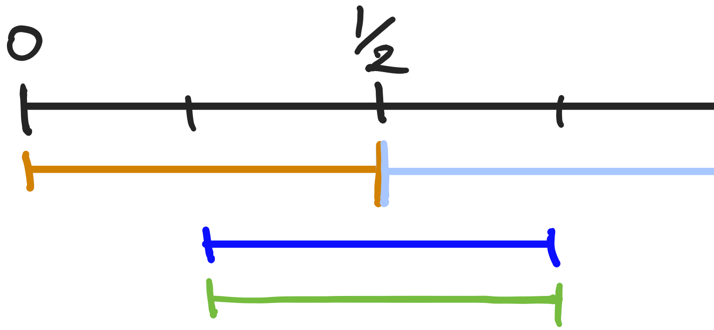

$$\mathcal{S}(\mathcal{A}) = \lambda(\mathcal{A}) = \mathrm{LOG}(L/M)$$