Overlapping random self-similar sets on the line

Vilma Orgoványi
joint work with Károly Simon

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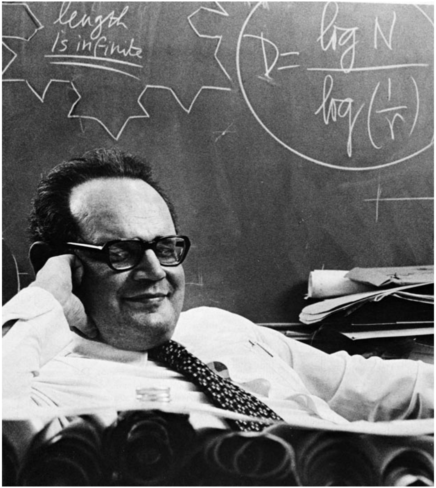
Benoit Mandelbrot



Construction of the (homogeneous) Mandelbrot percolation fractal $\Lambda_d(M, p)$ in $\mathbb{R}^d$

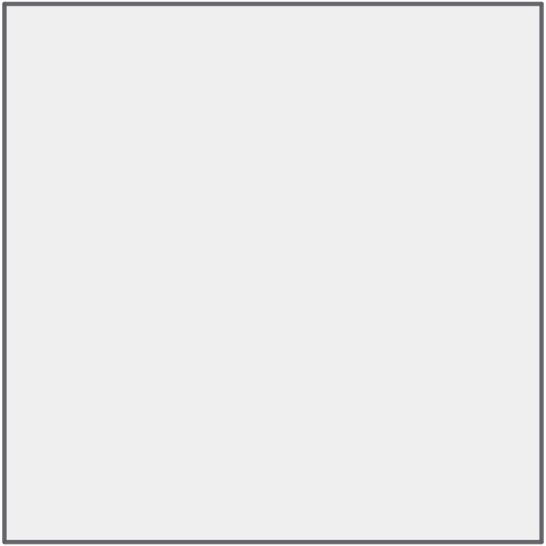
- $M \in \mathbb{N} \setminus \{0, 1\}$: division parameter
- $p \in (0, 1)$: probability

$\mathbb{P}($ 

$) = p$ and $\mathbb{P}($ 

$) = 1 - p$.

Construction



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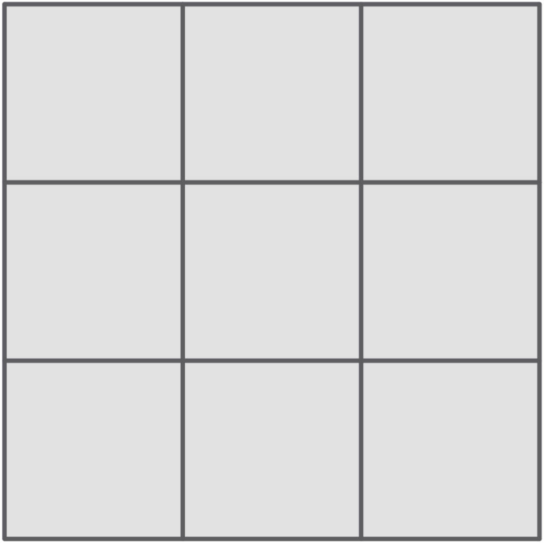

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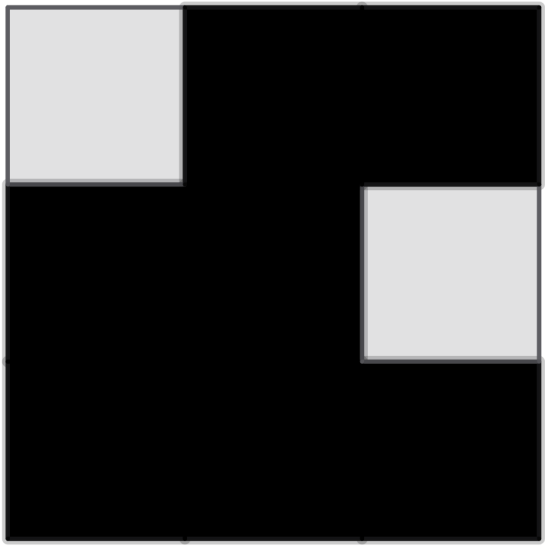

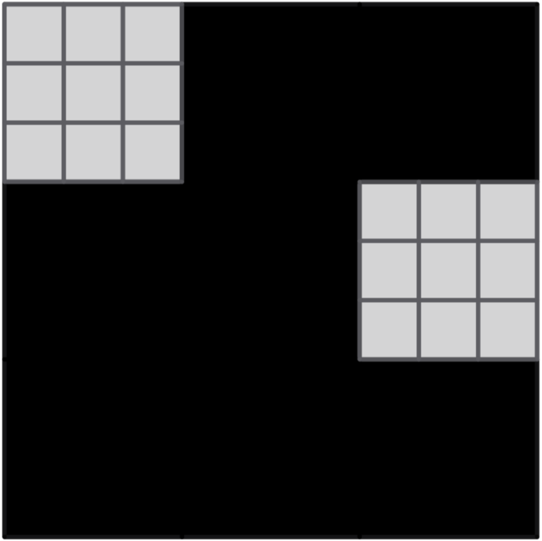

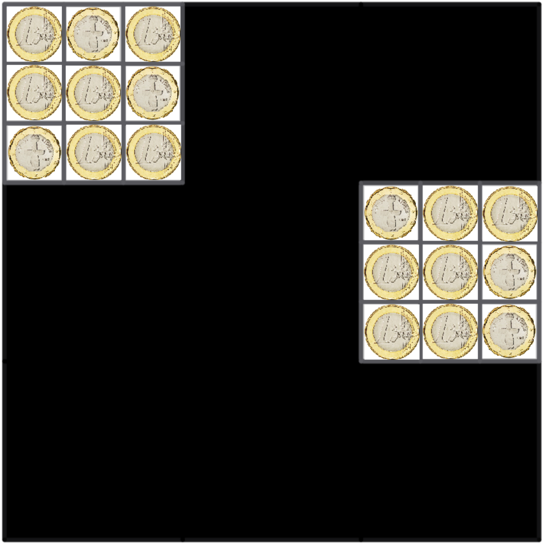

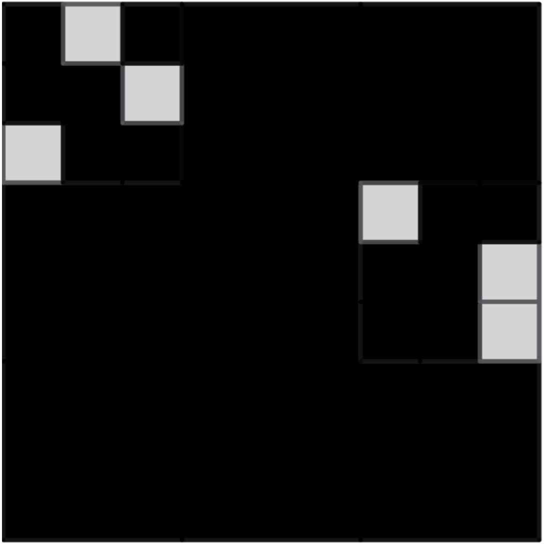

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The resulting set is $\Lambda_{M,p}$.

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2. Falconer, Mauldin-Williams: $\dim_H \Lambda_{M,p} = \frac{\log M^2 p}{\log M}$ a.s. conditioned on non-extinction;
3. Simon-Rams (2-dim), Simon-Vágó (d-dim): If $\dim_H \Lambda_{M,p} > 1$, then for almost all realizations (conditioned on non-extinction) simultaneously to all lines of $\mathbb{R}^d$ the orthogonal projection contains an interval.

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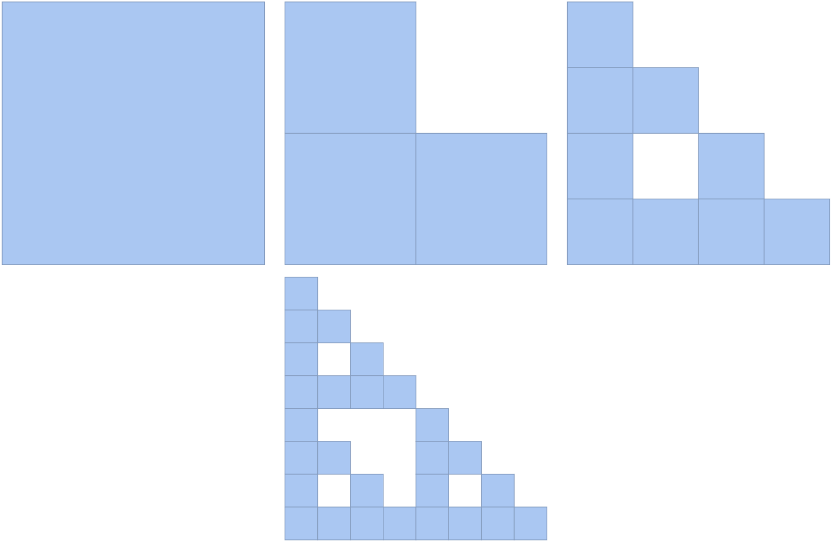
In particular, if $M = 3$

- $p > \frac{1}{9}$ $\Lambda_{3,p} \neq \emptyset$ with positive probability;
- $p > \frac{1}{3}$ if we exclude a set of realizations of extinction and further a realizations of 0 measure, for the remaining set of realizations the projection to every line contains an interval.

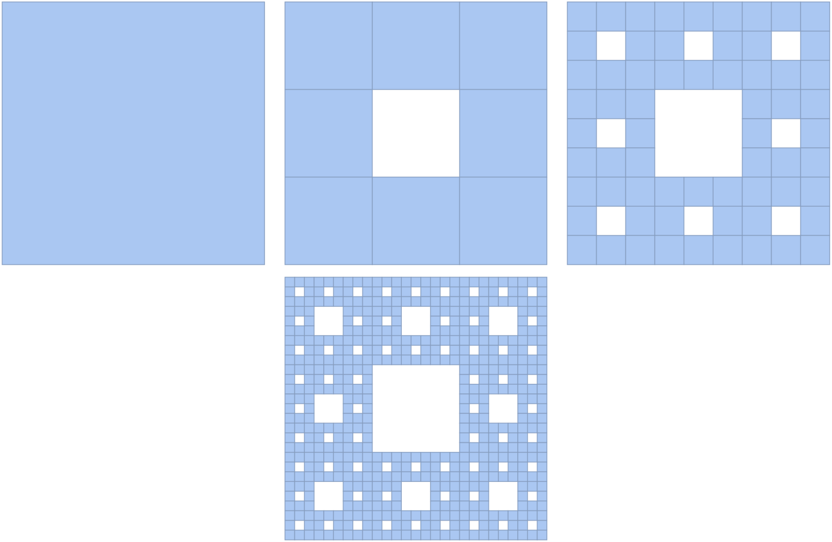
Homogeneous and inhomogeneous Mandelbrot percolation

We call the Mandelbrot percolation introduced above homogeneous Mandelbrot percolation, where in level-n of the construction we divided each of the level-n retained cubes into M^d congruent subcubes and for each of these we tossed a coin to decide wether we retain it or not. As opposed to this in the case of the inhomogeneous Mandelbrot percolation , there are some preselected cubes that we always discard.

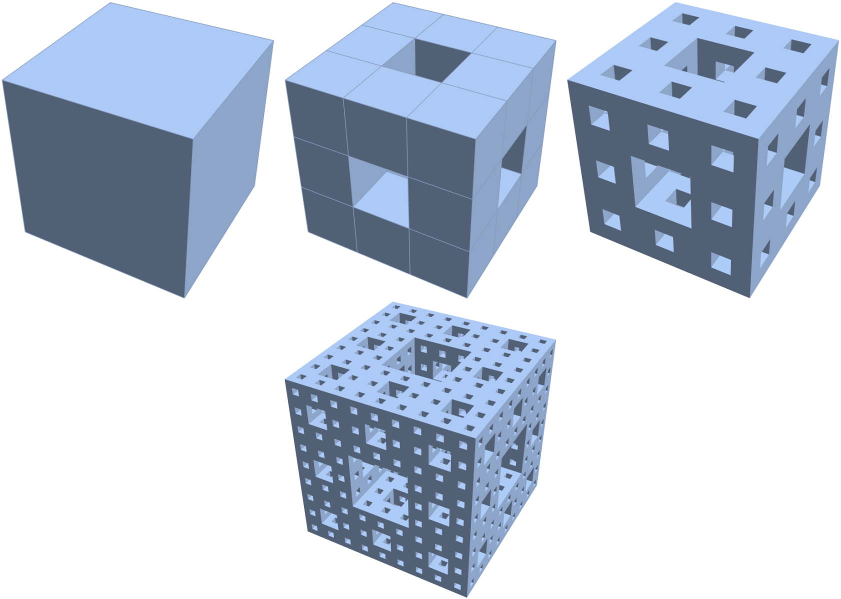
Right angled Sierpiński gasket



Sierpiński carpet



Menger sponge



Inhomogeneous Mandelbrot percolation

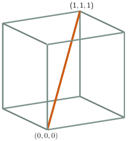
$\dim_H(\widetilde{\Lambda}_p) = \frac{\log(\mathbb{E}(\#\text{retained level 1 cubes}))}{-\log(\text{contraction ratio})}$ a.s. conditioned on non-extinction.

- Menger sponge: $\dim_{\mathrm{H}}(\mathcal{M}_p) = \frac{\log 20 \cdot p}{\log 3}$.
- Sierpiński carpet: $\dim_{\mathrm{H}}(\mathcal{S}_p) = \frac{\log 8 \cdot p}{\log 3}$.

1. Dekking-Grimmet (1988), Dekking-Meester (1989), Falconer (1989),Falconer-Grimmett (1992), Barral-Feng (2018): projections to the coordinate axes in the inhomogeneous case.
2. Simon and Vágó: rational projections of the random Sierpiński carpet.

Orthogonal projection of the random Menger sponge

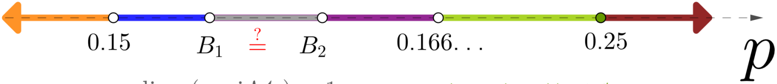
$\mathcal{M}_p$: random Menger sponge with parameter p;
proj: projection to the space diagonal of the unit cube;
$\text{proj}_{\underline{\alpha}}$: projection of the form $\underline{x} \to \underline{\alpha}\,\underline{x}$.



$\dim_H(\mathcal{M}_p) > 1$ a.s.$*$
but
$\dim_H(\text{proj}(\mathcal{M}_p)) < 1$ a.s.

$\text{Int}(\text{proj}(\mathcal{M}_p)) = \emptyset$ a.s.
but
$\mathcal{L}eb_1(\text{proj}(\mathcal{M}_p)) > 0$ a.s.$*$

$\forall\, \underline{\alpha}\ \text{Int}(\text{proj}_{\underline{\alpha}}(\mathcal{M}_p)) \neq \emptyset$ a.s.$*$



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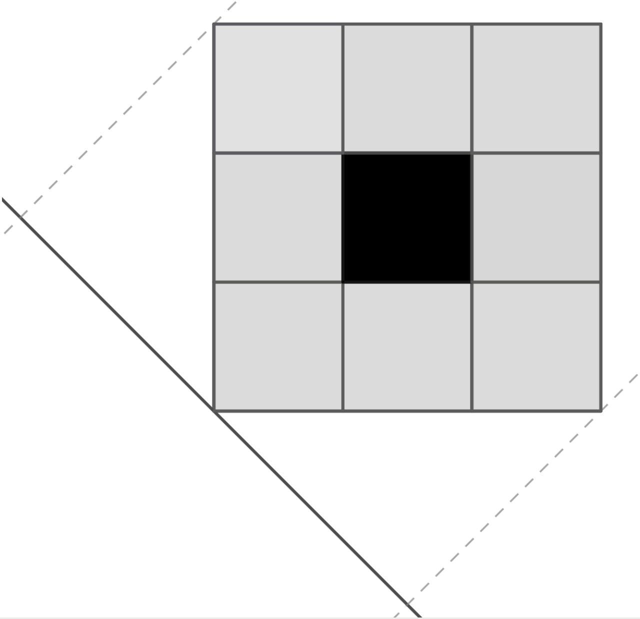
$\text{Int}(\text{proj}(\mathcal{M}_p)) \neq \emptyset$ a.s.$*$

$*$=conditioned on non-extinction

$0.15 < B_2 < 0.1514\ldots \quad 0.15 = \frac{3}{20}, \quad 0.166\cdots = \frac{1}{6}, \quad 0.25 = \frac{1}{4}.$

Construction of the matrices



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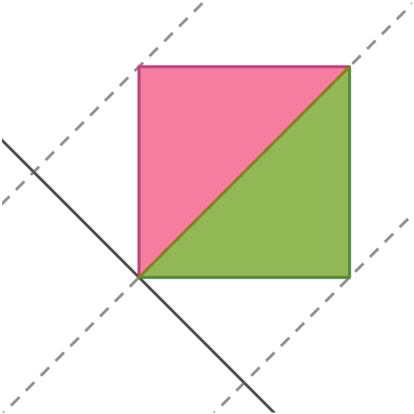

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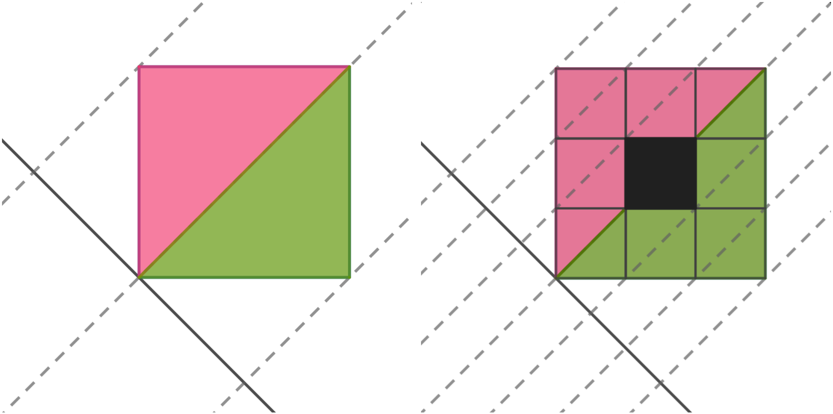

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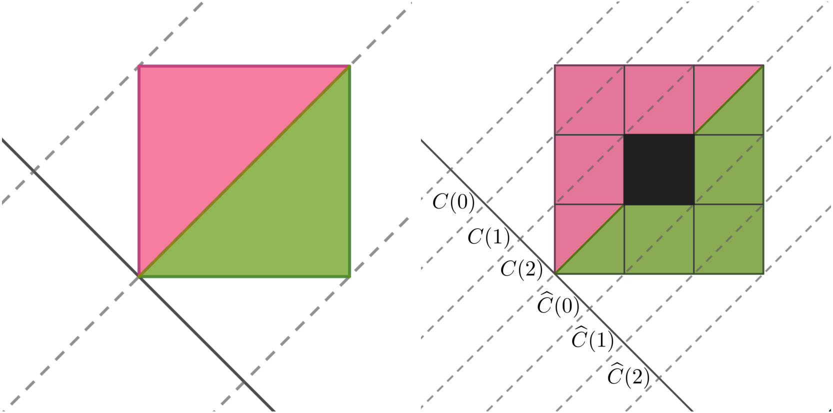

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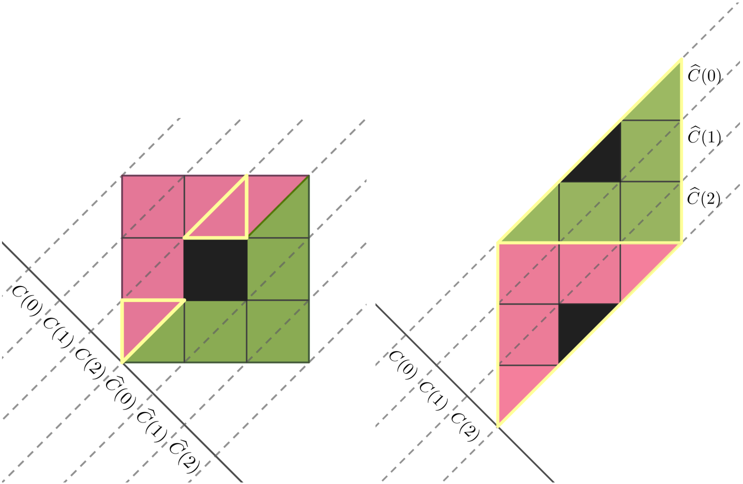

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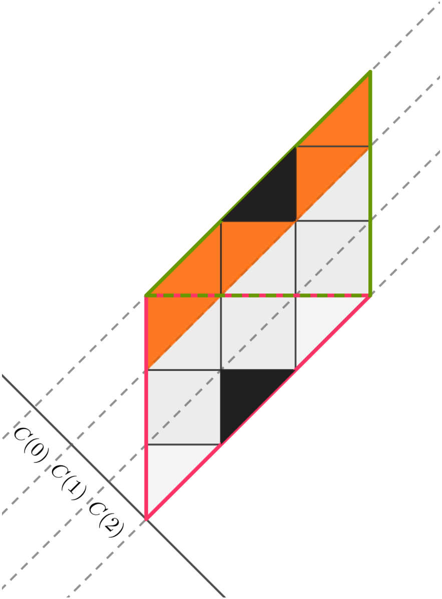

$$A_0 = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

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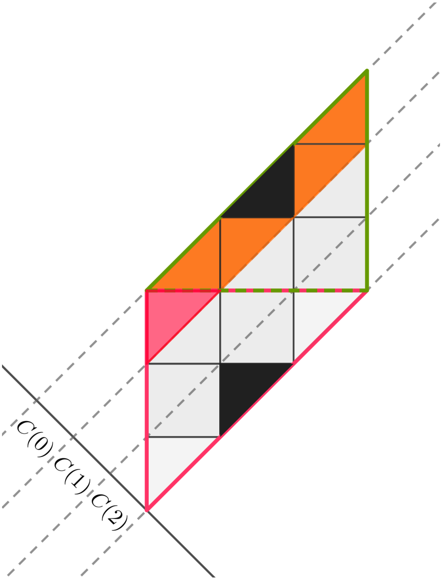

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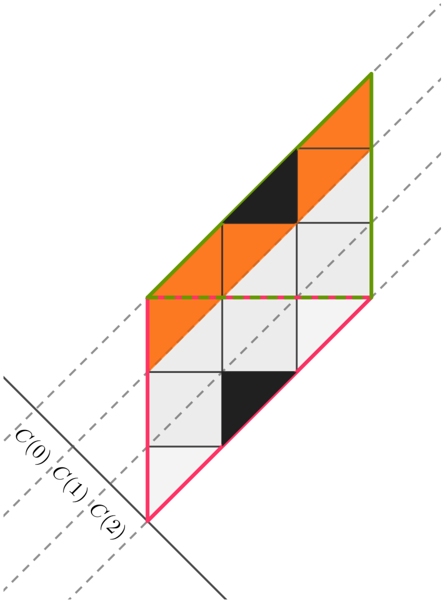

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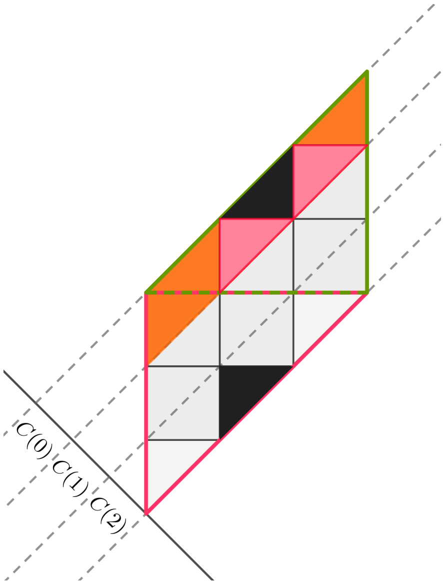

$$A_0 = \begin{pmatrix} 1 & 0 \\ 2 & x \end{pmatrix}$$

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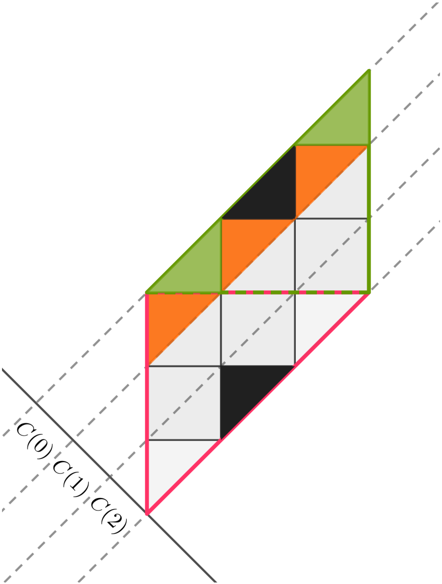

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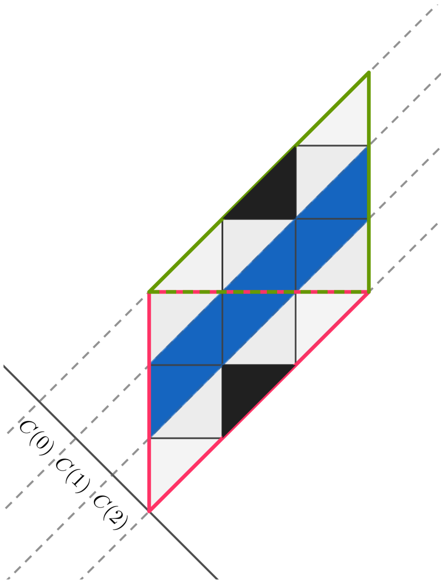

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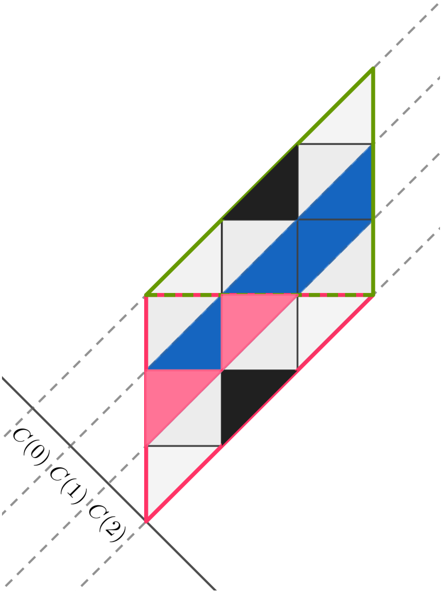

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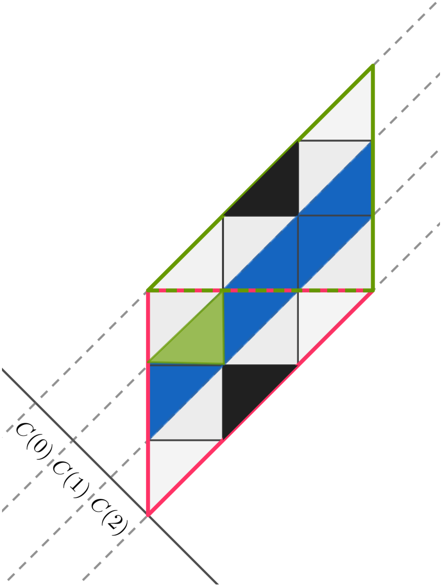

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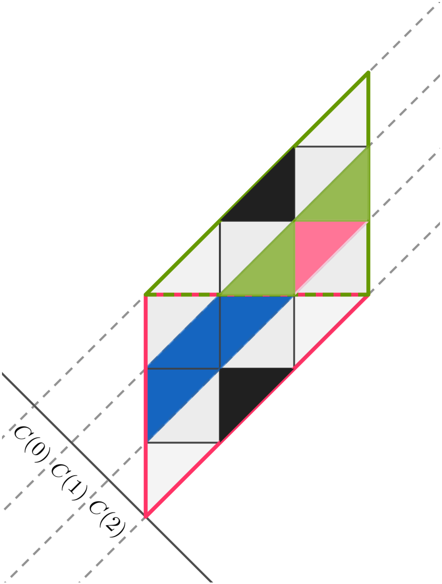

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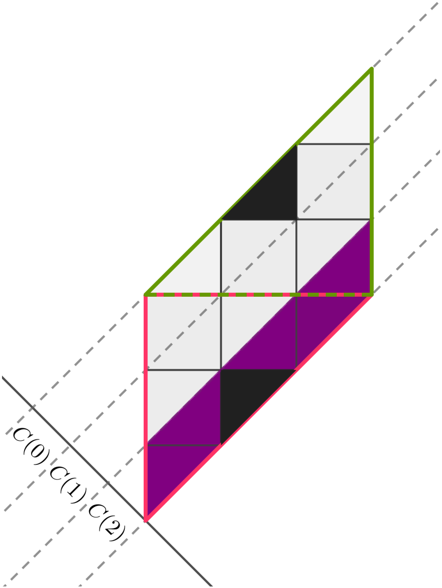

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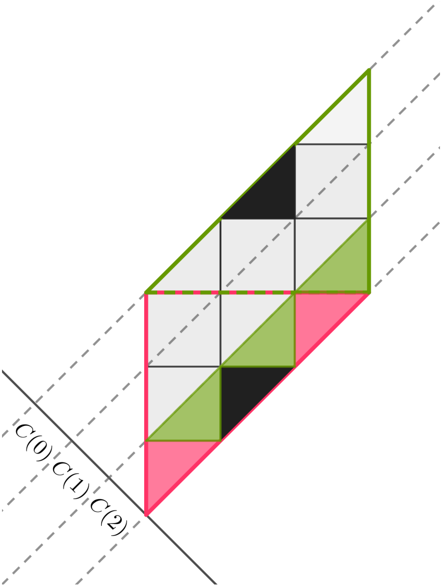

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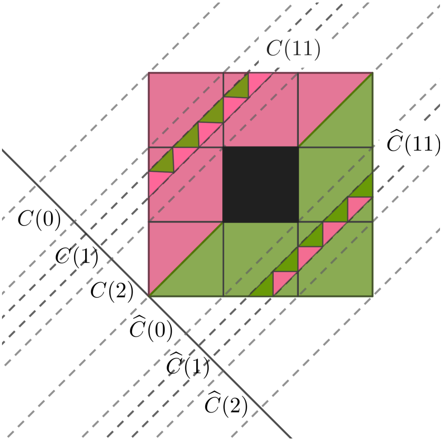

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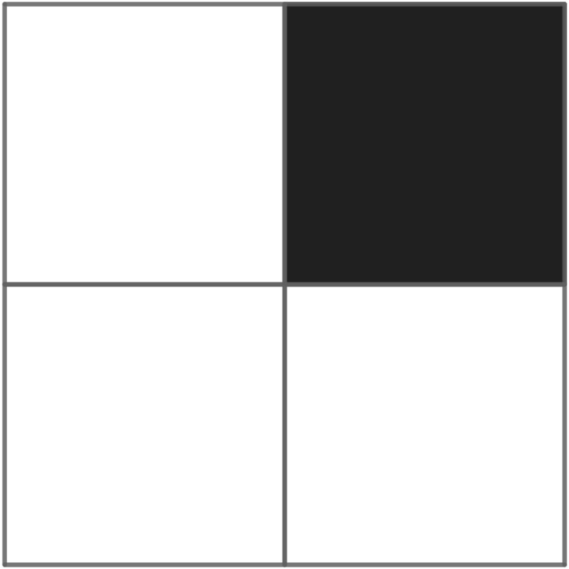
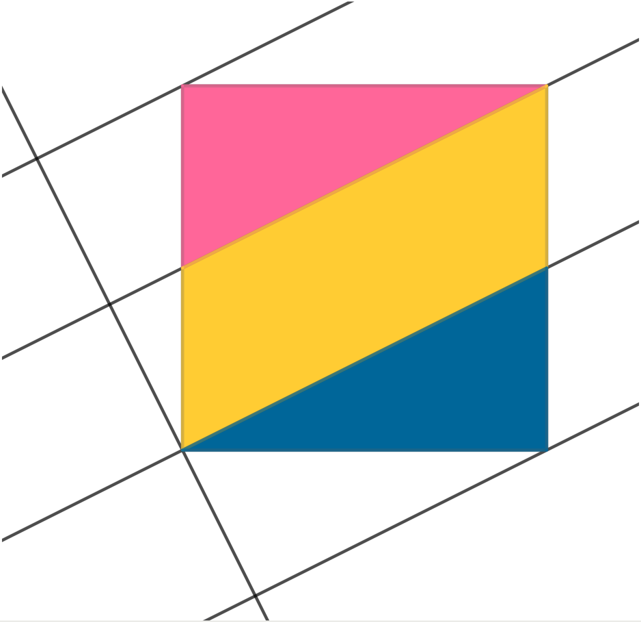
$$A_1 \cdot A_1 = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

Construction of the matrices 2.



Construction of the matrices 2.



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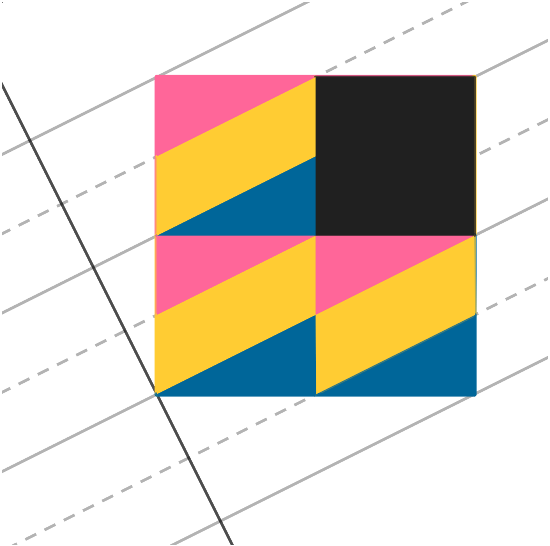

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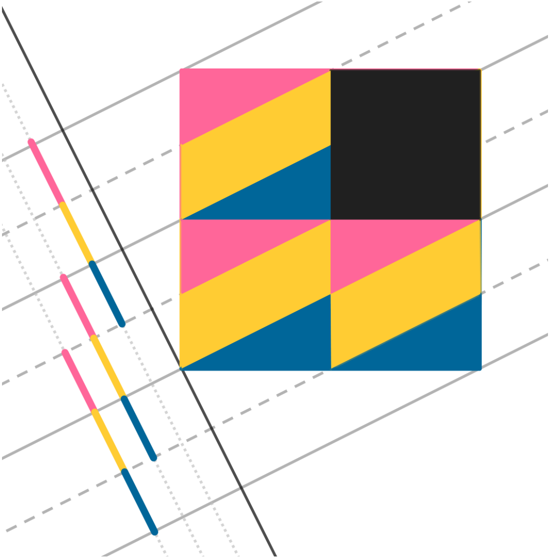

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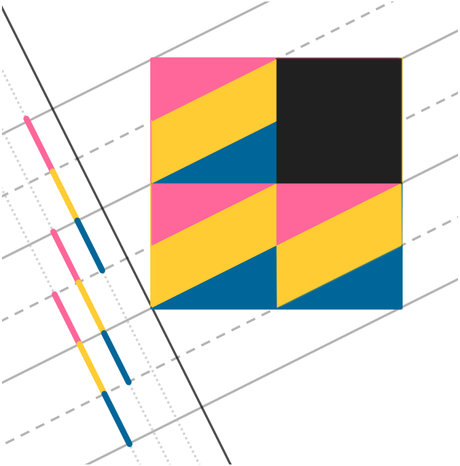

$$\mathcal{S} = \{S_i(x) = \frac{1}{L}x + t_i\}_{i=0}^{M-1},$$

- $L \in \mathbb{N} \setminus \{0, 1\}$,
- $t_i \in \mathbb{Q}$.

Lyapunov exponent and Lower spectral radius

For $\mathcal{A} = \{A_0, \dots, A_{L-1}\}$

- $\Sigma := \{0, \dots, L-1\}^{\mathbb{N}}$
- $\mu := \left(\frac{1}{L}, \dots, \frac{1}{L}\right)^{\mathbb{N}}$.
- $\|\cdot\|$ denote a submultiplicative matrix norm.

Definition (The Lyapunov exponent of $\mathcal{A}$)

$$\lambda(\mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} \log \|A_{i_1} \cdots A_{i_n}\| \text{ for } \mu \text{ a.e. } (i_1, \dots, i_n, \dots)$$

Definition (The Lower spectral radius of $\mathcal{A}$)

$$\underline{\rho}(\mathcal{A}) := \lim_{n\to\infty} \min\{\|A_{i_1} \cdots A_{i_n}\|^{1/n}, A_{i_j} \in \mathcal{A}\}$$

Positivity of Lebesgue measure

- $\mathcal{S} := \{S_i(x) = \frac{1}{L}x + t_i\}_{i=0}^{M}$, $t_i \in \mathbb{Q}$, $L \in \mathbb{N} - \{0, 1\}$.
- $\mathcal{A}_{\mathcal{S}} = \{A_0, \dots, A_{L-1}\}$, such that $\mathcal{A}_{\mathcal{S}}$ consists of allowable matrices and $\exists i_1, \dots, i_n \in [L]^n$ such that $A_{i_1} \dots A_{i_n}$ has only positive elements.
- Random attractor: $\Lambda_{\mathcal{S},p}$.

Theorem (Károly Simon, V.O.)

- *for $p > e^{-\lambda(\mathcal{A}_{\mathcal{S}})}$, $\mathcal{L}eb(\Lambda_{\mathcal{S},p}) > 0$ for almost every realization conditioned on non-extinction,*
- *for $p < e^{-\lambda(\mathcal{A}_{\mathcal{S}})}$, $\mathcal{L}eb(\Lambda_{\mathcal{S},p}) = 0$ almost surely.*

$\lambda(\mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} \log \|A_{i_1} \cdots A_{i_n}\|$ for μ a.e. $(i_1, \dots, i_n, \dots)$

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- *for* $p < e^{-\lambda(\mathcal{A})}$, $\mathcal{L}eb(\Lambda_{\mathcal{S},p}) = 0$ *almost surely.*

Checkable condition: Let $CS(i,j) = \sum_k A_i(k,j)$.
If $p > \max_j(\prod_i CS(i,j))^{-\frac{1}{L}}$, then $\mathcal{L}eb(\Lambda_{\mathcal{S},p}) > 0$ a.s. conditioned on non-extinction.

$\lambda(\mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} \log \|A_{i_1} \cdots A_{i_n}\|$ for μ a.e. $(i_1, \ldots, i_n, \ldots)$

Existence of interior points

- $\mathcal{S} := \{S_i(x) = \frac{1}{L}x + t_i\}_{i=0}^{M}$, $t_i \in \mathbb{Q}$, $L \in \mathbb{N} - \{0,1\}$.
- $\mathcal{A}_{\mathcal{S}} = \{A_0, \dots, A_{L-1}\}$, such that $\exists i_1, \dots, i_n \in [L]^n$ such that $A_{i_1} \dots A_{i_n}$ has a row with only positive elements,
- $CS(i,j) = \sum_k A_i(k,j)$.
- Random attractor: $\Lambda_{\mathcal{S},p}$.

Theorem (Károly Simon, V.O.)

- *for $p > (\min_{i,j} CS(i,j))^{-1}$, then $\Lambda_{\mathcal{S},p}$ contains an interval for almost every realization conditioned on non-extinction,*
- *for $p < \underline{\rho}(\mathcal{A})^{-1}$, then $\Lambda_{\mathcal{S},p}$ does not contain an interval almost surely.*

$\underline{\rho}(\mathcal{A}) := \lim_{n\to\infty} \min\{\|A_{i_1} \cdots A_{i_n}\|^{1/n}, A_{i_j} \in \mathcal{A}\}$

Thank you for your attention!