Dynamical systems, Spring 2020

Log: a brief summary of the classes

February 11

Dynamical system in discrete time: map. Orbit, periodicity, asymptotic behavior. Invertibility.

Further classification and connections depending on the preserved structure: topological dynamics, ergodic theory, smooth dynamics.

Dynamics in continuous time: flows, autonomous ODE systems.

Rotations of the circle. $\mathbb{S}^1$ as phase space. Invertibility, isometry, rigidity. Rational α: every point is periodic with the same period. Irrational α: every point has a dense orbit (definitions of topological transitivity and minimality) Lebesgue measure is invariant. Further invariant measures concentrated on periodic orbits in the rational case. Outlook: unique ergodicity for irrational α.

Doubling map or $2x \pmod 1$. Non-invertibility, expansion, dyadic rationals are eventually fixed.

February 17

One sided full shift with two symbols. $\Sigma^+ = \{0,1\}^{\mathbb{N}}$ as a compact space, metric, $\sigma : \Sigma^+ \to \Sigma^+$, the left shift.

Equivalence of dynamical systems. conjugacy, semi-conjugacy, factors. further issues: continuity, measurability, push forward of a measure.

The doubling map and the one sided full shift are (almost) conjugate, discussion of the conjugacy. Applications: characterization of periodic points and points with dense orbits. Invariance of Lebesgue measure for the doubling map. Cylinder sets, description of the $1/2 - 1/2$ Bernoulli measure, when pushed forward, gives Lebesgue. Many further invariant measures for the shift.

Further comments on the topology of the shift space: tryadic Cantor set.

February 24

Doubling map as $Tz = z^2$ on the complex unit circle. How $f : [-1,1] \to [-1,1]$, $f(x) = 2x^2 - 1$ is obtained as a factor, invariant density for the later. (Reminder: density transformation formula.)

Product of dynamical systems. Products of rotations.

Linear flow on $\mathbb{T}^2$.

Relating flows to maps and back: suspension flow and Poincaré section. Linear flow on the torus as a suspension of a rotation.

Linear self-maps of the real line. Continuous maps of the real line: graphical analysis (cobweb plot), attracting and repelling fixed points.

The logistic family $T_\mu x = \mu x(1-x)$, motivation from population dynamics.

February 25

Fixed points for the logistic family: the case $\mu \leq 1$. $\mu > 1$: orbits of $x \notin [0,1]$.

Analysis reminder: intermediate value theorem, mean value theorem, implicit function theorem.

A fixed point cannot disappear or split unless $f'(x_0) = 1$.

Logistic family: description of the attracting fixed point for $1 < \mu \leq 2$ and $2 < \mu < 3$.

$\mu = 3$: discussion of the second iterate, inflection point, period-doubling bifurcation.

Illustration for the complexity of $3 < \mu < 4$.

Discussion of logistic maps with $\mu > 4$ (for simplicity restrict to $\mu > 2 + \sqrt{5}$).

March 3

Logistic maps with $\mu > 2 + \sqrt{5}$: intervals I_0 and I_1, inverse branches, construction of the invariant Cantor set. Topological conjugacy with the one-sided shift. Repeller.

Saddle-node bifurcation in one dimension.

Periodic points for continuous maps $T : \mathbb{R} \to \mathbb{R}$. Existence of a period 3 orbit implies existence of periodic orbits with (least) period n for every $n \in \mathbb{N}$. Statement of Sharkovsky's theorem (without proof).

C^r metrics. Structural stability. $\frac{1}{2}x$ is C^1-structurally stable, logistic map with $\mu > 2 + \sqrt{5}$ is C^2-structurally stable. Hartman's theorem on the structural stability of hyperbolic fixed points.

March 5

Gauss map. Connection to continued fraction expansions. Rational points are eventually fixed. The golden mean as a fixed point of the Gauss map. Perron-Frobenius operator. Invariant density for the Gauss map.

Linear maps of the plane. Reminder: Jordan canonical form for real matrices. How the phase portrait is determined by the spectrum: source, sink, saddle, focus, node. Verifying stability by Lyapunov functions. Stable and unstable subspaces.

Two dimensional nonlinear maps: behavior in the vicinity of a fixed point. Hyperbolic fixed points: Hartman-Grobman theorem.