Dynamical systems, Spring 2022

Homework problem set #1 . Due on March 29, Tuesday

One of the 10 problems below can be regarded as a bonus problem. That is, with complete solutions for 9 problems you can obtain full credit. Solving all the 10 problems properly deserves extra credit.

1. Consider the one dimensional map $T_\lambda : \mathbb{R} \to \mathbb{R}$, $T_\lambda x = x^5 - \lambda x$, where the parameter λ satisfies $-\infty < \lambda \leq 1$. Investigate the λ-dependence of
   (a) the fixed points and their stability properties.
   (b) the asymptotic behavior of the orbit $T_\lambda^n x$, $n \geq 0$ for any initial condition $x \in \mathbb{R}$.
2. Consider the doubling map $T : \mathbb{S}^1 \to \mathbb{S}^1$, $Tx = 2x \pmod 1$. Let $D \subset \mathbb{S}^1$ denote the set of points x such that $\{T^n x | n = 0, 1, 2 \ldots\}$ is dense in $\mathbb{S}^1$. Prove that $\lambda(D) = 1$ (where λ is the Lebesgue measure).
3. Consider $T : [0, 1] \to [0, 1]$, $Tx = 4x(1 - x)$ (the logistic map with $\mu = 4$) . Verify that T has an absolutely continuous invariant (probability) measure, with density $\rho(x) = C(x(1 - x))^{-1/2}$. ($C =?$)
4. Fix a positive integer $K \geq 1$, and let $\Sigma_K^+ = \{0, 1, ..., K - 1\}^{\mathbb{N}}$, that is, the space of infinite sequences of K symbols. The shift map $\sigma : \Sigma_K^+ \to \Sigma_K^+$ is defined in the usual way.
   (a) What are the periodic points of this shift map with K symbols? Specify a point in Σ_K^+ that has a dense orbit.
   (b) Consider now the map $T_K : \mathbb{S}^1 \to \mathbb{S}^1$, $T_K x = Kx \pmod 1$. How are these two dynamical systems related?
5. Recall from class that on $\Sigma^+ = \{0, 1\}^{\mathbb{N}}$, that is, the space of infinite sequences of 2 symbols, we defined a metric as $d(\underline{a}, \underline{b}) = 2^{-s(\underline{a},\underline{b})}$, where $s(\underline{a}, \underline{b}) = \min\{k \geq 1 | a_k \neq b_k\}$ (here $\underline{a} = (a_1, a_2, ...)$ and similarly for $\underline{b}$). Prove that $d(\underline{a}, \underline{b})$ is indeed a metric, in particular, it satisfies the triangular inequality.
6. Show that the map $T : [-1, 1] \to [-1, 1]$, $Tx = 8x^4 - 8x^2 + 1$ has an absolutely continuous invariant (probability) measure, and determine the density. (*Hint*: in class we discussed the case of $Tx = 2x^2 - 1$, you may proceed along the same lines, just instead of $2x \pmod 1$ consider $4x \pmod 1$ as a map of the unit circle in $\mathbb{C}$.)
7. Consider the linear maps $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x) = Ax$ for the matrices A below. In each case, describe the asymptotic behavior and sketch the phase portrait. In hyperbolic cases, determine the stable and unstable subspaces (W^s and W^u).

   (a) $\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 1 \\ 0 & 1/3 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1/5 \\ 1/5 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}$.
8. Let $T : \mathbb{T}^2 \to \mathbb{T}^2$ be a hyperbolic toral automorphism. Show that, for the matrix A associated to T, the eigenvalues are irrational numbers while the eigendirections, as lines on $\mathbb{R}^2$, have irrational slope.
9. Let $T : \mathbb{T}^2 \to \mathbb{T}^2$ be a hyperbolic toral automorphism. Show that $x \in \mathbb{T}^2$ is a periodic point for T if and only if both of its coordinates are rational.
10. Consider the logistic map $T_\mu x = \mu x(1 - x)$ with $\mu > 2 + \sqrt{5}$. Show that there exists some $\lambda > 1$ such that $|T_\mu'(x)| > \lambda$ whenever $x \in I_0 \cup I_1$. (Recall that we denoted $T_\mu^{-1}[0, 1] = I_0 \cup I_1$, where I_0 and I_1 are disjoint intervals.)