Dynamical systems, Spring 2022

Homework problem set #2 . Due on May 3, Tuesday

One of the 8 problems below can be regarded as a bonus problem. That is, with complete solutions for 7 problems you can obtain full credit. Solving all the 8 problems properly deserves extra credit.

1. If $T : M \to M$ is a dynamical system with invariant measure μ and $n \geq 2$ is a fixed integer, then it is possible to consider $T^n = T \circ \cdots \circ T$, the nth power of T, for which μ is again invariant. Show that whenever T^n is ergodic, then T is ergodic. On the other hand, the converse is not true; give an example when T is ergodic but T^n (for some n fixed) is not ergodic.

2. Consider the rotation $T : \mathbb{S}^1 \to \mathbb{S}^1$, $Tx = x + 1/2 \pmod 1$. Verify that the following two measures are invariant:
$$\mu_1 = \frac{1}{2}\left(\delta_{\frac{1}{4}} + \delta_{\frac{3}{4}}\right); \qquad \mu_2 = \frac{1}{4}\left(\delta_0 + \delta_{\frac{1}{4}} + \delta_{\frac{1}{2}} + \delta_{\frac{3}{4}}\right).$$
Are μ_1 and/or μ_2 ergodic? Are they mixing (with respect to T)?

3. Let M be a compact metric space, $T : M \to M$ continuous, and let $\mathcal{M}_{inv}$ denote the collection of T-invariant Borel probability measures, while $\mathcal{M}_{erg} \subset \mathcal{M}_{inv}$ is the subcollection of ergodic measures. Show that whenever $\mu \in \mathcal{M}_{inv}$ but $\mu \notin \mathcal{M}_{erg}$, μ is *not* an extreme point of the convex set $\mathcal{M}_{inv}$, that is, there exist $\mu_1, \mu_2 \in \mathcal{M}_{inv}$ and $0 < t < 1$ such that $\mu = t\mu_1 + (1-t)\mu_2$.

4. Consider $T : [0,1] \to [0,1]$, $Tx = x + a\sin(2\pi x)$, where $0 < a < \frac{1}{2\pi}$. Describe $\mathcal{M}_{inv}$ and $\mathcal{M}_{erg}$ for this map.

5. Show that the conditions of the Krylov-Bogolyubov theorem are „sharp" by giving examples of maps with $\mathcal{M}_{inv} = \emptyset$ and
   (a) $T : [0,1] \to [0,1]$, but T is not continuous (minimize the number of discontinuity points);
   (b) $T : (0,1) \to (0,1)$ and T is continuous;
   (c) $T : \mathbb{R} \to \mathbb{R}$ and T is continuous.

6. Consider the first two bits in the binary expansions of the numbers $3, 9, 27, \ldots, 3^n, \ldots$. What is more frequent, 11 or 10?

7. Piecewise linear (strong) Markov maps $T : [0,1] \to [0,1]$ are associated to finite partitions $0 = a_0 < a_1 < a_2 < \cdots < a_{K-1} < a_K = 1$ of the interval $[0,1]$. When restricted to the subinterval $I_j = (a_{j-1}, a_j)$, the map is given by $T_j := T|_{I_j}$; $T_j x = \frac{x - a_{j-1}}{a_j - a_{j-1}}$, that is, T_j is a *linear* one-to-one map of I_j onto $(0,1)$ ($j = 1, 2, \ldots K$).
   (a) Show that the Lebesgue measure is invariant for T.
   (b) Show that T with the Lebesgue measure is isomorphic to a Bernoulli shift with K symbols (the probability distribution on the K symbols has to be defined appropriately, of course). Conclude that T is ergodic (in fact, mixing) with respect to the Lebesgue measure.
   (c) For (Lebesgue) almost every $x \in (0,1)$ the quantity $|(T^n)'(x)|$ – that is, the derivative of the nth iterate – is well-defined. Show that $|(T^n)'(x)|$ grows exponentially with the same rate λ for almost every $x \in (0,1)$. Remark: in this simple context, λ is the occurrence of the „asymptotic expansion rate" or „Lyapunov-exponent". Recall that a numerical sequence b_n grows exponentially with rate λ if $\lim_{n\to\infty} \frac{\ln b_n}{n} = \lambda$.

8. Let M be a topological space and $T : M \to M$ a continuous map.

- For $x \in M$ the ω-*limit* points of x are

$$\omega(x) = \bigcap_{n \in \mathbb{Z}^+} \overline{\bigcup_{j \geq n} T^j x};$$

  that is $y \in \omega(x)$ if and only if there is a subsequence $n_k \to \infty$ such that $T^{n_k} x \to y$.

- The *recurrent points* of T are:

$$\mathcal{R}(T) = \{x \in M \mid x \in \omega(x)\}.$$

- $x \in M$ is a *non-wandering* point for T if given any open neighborhood $U \ni x$ there exists $n \geq 1$ such that $T^n U \cap U \neq \emptyset$. The collection of non-wandering points is denoted by $\Omega(T)$.

Show that (i) $\Omega(T)$ is closed; (ii) if $y \in \Omega(T)$ then $Ty \in \Omega(T)$; (iii) for any $x \in M$ it holds that $\omega(x) \subset \Omega(T)$; hence (iv) $\overline{\mathcal{R}(T)} \subset \Omega(T)$; and we have the following chain of implications (v) x is periodic $\Rightarrow$ x is recurrent $\Rightarrow$ x is non-wandering; however (vi) give examples that all these inclusions are proper.