Dynamical systems, Spring 2022

Homework problem set #3. Due on May 24, Tuesday

One of the 6 problems below can be regarded as a bonus problem. That is, with complete solutions for 5 problems you can obtain full credit. Solving all the 6 problems properly deserves extra credit.

1. We discussed in class that $\Lambda \subset M$ is an *attractor* for the invertible topological dynamical system $T : M \to M$ if there exists an open neighborhood $U \supset \Lambda$ such that for the closure $N = \overline{U}$ it holds that $T(N) \subset U$ (that is, U is a *trapping region*) and $\Lambda = \bigcap_{n=0}^{\infty} T^n N$. Show that in such a case Λ is a closed invariant set (invariance means $\Lambda = T(\Lambda) = T^{-1}(\Lambda)$).
2. As usual, let us represent $\mathbb{T}^2$ as $[0,1]^2$ with the opposite sides identified. Let $a > 0$ be some small parameter (say $a < \frac{1}{100}$) and consider $F : \mathbb{T}^2 \to \mathbb{T}^2$, $F(x,y) = (x + a\sin(2\pi x), y + a\cos(2\pi x)\sin(2\pi y))$. Identify the fixed points, verify that all of them are hyperbolic, and classify them into sources, sinks and saddles. Find the (global) stable and unstable manifolds for each fixed point.
3. Consider the one-sided full shift with two symbols, $\sigma : \Sigma^+ \to \Sigma^+$. Prove that this system has the shadowing property. That is, given a δ-pseudo orbit construct a point the true orbit of which is ε-shadowing the pseudo orbit (where δ is appropriately chosen for ε).
4. Let M be a compact metric space, and for $\varepsilon > 0$ let $C(\varepsilon)$, $N(\varepsilon)$ and $S(\varepsilon)$ denote the minimum cardinality of ε-covers, the minimum cardinality of ε-nets, and the maximum cardinality of ε-separated sets, respectively. Prove that $C(2\varepsilon) \le N(\varepsilon) \le S(\varepsilon) \le C(\varepsilon)$.
5. Prove (ii), (iii), (iv), (vi), (vii) and (viii) from the list of properties of $H(\alpha)$ and $H(\alpha|\beta)$. (You may rely on (i) and (v).)
6. Prove (1), (2), (3), (5), (6) and (7) from the list of properties of $h(T)$. (You may rely on (4) and (i-xii).)