Dynamical systems, Sample Final Exam, May 35, 2022; 5.10am – 6.40

Show your work! Provide a clear, complete and detailed reasoning for each of the problems!

1. (a) Define the notions of a δ-chain (or δ-pseudo orbit) and of ε–shadowing. Using these, define what it means that a dynamical system satisfies the shadowing property! (6 points)

   (b) Prove that the doubling map $T : \mathbb{S}^1 \to \mathbb{S}^1$, $Tx = 2x \ (mod\ 1)$ satisfies the shadowing property. (14 points)

2. (a) What do we mean by a transition matrix (π_{ij}), and an adjacency matrix (A_{ij}), and when do we say that these are primitive (or irreducible and aperiodic)? Provide the definition of the associated topological Markov chain and of the associated Markov shift. (6 points)

   (b) Prove that the associated Markov shift is mixing. (14 points)

3. Consider the rotation $T : \mathbb{S}^1 \to \mathbb{S}^1$, $Tx = x + 1/2$ (mod 1). Verify that the following two measures are invariant:
$$\mu_1 = \frac{1}{2}\left(\delta_{\frac{1}{3}} + \delta_{\frac{5}{6}}\right); \qquad \mu_2 = \frac{1}{4}\left(\delta_{\frac{1}{6}} + \delta_{\frac{1}{3}} + \delta_{\frac{2}{3}} + \delta_{\frac{5}{6}}\right).$$
Are μ_1 and/or μ_2 ergodic? Are they mixing (with respect to T)? (20 points)

4. Consider the linear maps $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x) = Ax$ for the matrices A below. In each case, describe the asymptotic behavior and sketch the phase portrait. In hyperbolic cases, determine the stable and unstable subspaces (W^s and W^u). (20 points)

   (a) $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1/3 & 1 \\ 0 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 2\sqrt{3} \\ -2\sqrt{3} & 2 \end{bmatrix}$.