Dynamical systems, Spring 2022

Log: a brief summary of the classes

February 14

Dynamical system in discrete time: map. Orbit, periodicity, asymptotic behavior. Invertibility.

Further classification and connections depending on the preserved structure: topological dynamics, ergodic theory, smooth dynamics.

Dynamics in continuous time: flows, autonomous ODE systems.

Rotations of the circle. $\mathbb{S}^1$ as phase space. Invertibility, isometry, rigidity. Rational α: every point is periodic with the same period. Irrational α: every point has a dense orbit (definitions of topological transitivity and minimality) Lebesgue measure is invariant. Further invariant measures concentrated on periodic orbits in the rational case. Outlook: unique ergodicity for irrational α.

February 15

Doubling map or $2x \pmod 1$. Non-invertibility, expansion, dyadic rationals are eventually fixed.

One sided full shift with two symbols. $\Sigma^+ = \{0,1\}^{\mathbb{N}}$ as a compact space, metric, $\sigma : \Sigma^+ \to \Sigma^+$, the left shift.

Equivalence of dynamical systems. conjugacy. Further issues: continuity, measurability, push forward of a measure.

The doubling map and the one sided full shift are (almost) conjugate, discussion of the conjugacy. Applications: characterization of periodic points and points with dense orbits. Invariance of Lebesgue measure for the doubling map. Cylinder sets, description of the $1/2 - 1/2$ Bernoulli measure, when pushed forward, gives Lebesgue.

Further comments on the topology of the shift space: tryadic Cantor set.

Many further invariant measures for the shift.

February 21

Semi-conjugacy, factors.

Doubling map as $Tz = z^2$ on the complex unit circle. How $f : [-1,1] \to [-1,1]$, $f(x) = 2x^2 - 1$ is obtained as a factor, invariant density for the later. (Reminder: density transformation formula.)

Products of dynamical systems: products of two rotations as a map of the torus $\mathbb{T}^2$.

Linear flow on $\mathbb{T}^2$.

Relating flows to maps and back: suspension flow and Poincaré section. Linear flow on the torus as a suspension of a rotation.

Linear self-maps of the real line. Continuous maps of the real line: graphical analysis (cobweb plot), attracting and repelling fixed points.

The logistic family $T_\mu x = \mu x(1-x)$, motivation from population dynamics. Fixed points: the case $\mu \le 1$. $\mu > 1$: orbits of $x \notin [0,1]$.

February 22

Analysis reminder: intermediate value theorem, mean value theorem, implicit function theorem.

A fixed point cannot disappear or split unless $f'(x_0) = 1$.

Logistic family: description of the attracting fixed point for $1 < \mu \le 2$ and $2 < \mu < 3$.

$\mu = 3$: discussion of the second iterate, inflection point, period-doubling bifurcation.

Illustration for the complexity of $3 < \mu < 4$.

Discussion of logistic maps with $\mu > 4$ (for simplicity restrict to $\mu > 2 + \sqrt{5}$). Intervals I_0 and I_1, inverse branches, construction of the invariant Cantor set. Topological conjugacy with the one-sided shift. Repeller.

February 28

Saddle-node bifurcation in one dimension.

Periodic points for continuous maps $T : \mathbb{R} \to \mathbb{R}$. Existence of a period 3 orbit implies existence of periodic orbits with (least) period n for every $n \in \mathbb{N}$. Statement of Sharkovsky's theorem (without proof).

March 1

C^r metrics. Structural stability. $\frac{1}{2}x$ is C^1-structurally stable, logistic map with $\mu > 2+\sqrt{5}$ is C^2-structurally stable.

Gauss map. Connection to continued fraction expansions. Rational points are eventually fixed. The golden mean as a fixed point of the Gauss map.

Perron-Frobenius operator. Invariant density for the Gauss map.

Linear maps of the plane. Reminder: Jordan canonical form for real matrices.

March 7

How the phase portrait is determined by the spectrum: source, sink, saddle, focus, node. Verifying stability by Lyapunov functions. Stable and unstable subspaces.

Two dimensional nonlinear maps: behavior in the vicinity of a fixed point. Hyperbolic fixed points: Hartman-Grobman theorem. Hyperbolicity of the fixed point is essential: examples of non-hyperbolic fixed points with attracting/repelling behavior. Hopf bifurcation.

March 8

Toral automorphisms, definitions, invertibility. Discussion of an elliptic ($\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$) and a parabolic ($\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$) example.

Hyperbolic toral automorphisms, discussed via the particular example of the CAT map $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$: sketch of domains, description of periodic points. Stable and unstable manifolds and foliations. Backward and forward asymptotic points, consequences of the irrational slope. Existence of a dense set of points homoclinic to the origin.

March 21

Two equivalent characterizations of topological transitivity (Baire category theorem).

Hyperbolic toral automorphisms are topologically transitive; proof based on the density of homoclinic points.

Smale's horseshoe map. The sets $H_0, H_1, V_0, V_1, H_{00}, H_{10}, H_{11}, H_{01}$ etc. $\Lambda = \Lambda^+ \cap \Lambda^- = \Lambda_1 \times \Lambda_2$, as the product of two Cantor sets.

March 22

Double sided full shift: product topology, separation metric, invertibility. Topological conjugacy with the Smale horseshoe. Periodic points, stable and unstable sets of a point, homoclinic points – geometric and symbolic characterization.

Ergodic theory. Introductory examples: three piecewise linear maps with constant slope 2. Scope of ergodic theory: invariant measures and their properties. Invariant sets and invariant functions, corresponding equivalent definitions of ergodicity.

Discussion of Homework #1.

March 26

Discussion of Homework #1 continued.

$T : M \to M$ with invariant measure μ, associated linear isometry $\hat{T}$ on the Banach spaces L^p_μ. The case of the Hilbert space L^2_μ; definition of $\hat{T}^*$, $f = \hat{T}^* f \Leftrightarrow f = \hat{T} f$.

Convergence almost everywhere and in L^p. Birkhoff's and von Neumann's ergodic theorems. Proof of von Neumann's ergodic theorem. Discussion of the ergodic case: strong law of large numbers. Review of the introductory examples from the viewpoint of ergodicity.

March 28

X compact metric space: $C(X)$ Banach space of continuous functions (sup norm), $\mathcal{M}$ collections of Borel measures on X, as a subset of $C^*(X)$.

Borel measures as positive bounded linear functionals, Riesz representation theorem. Weak-$*$ topology, Banach-Alaoglu theorem.

$T : X \to X$ continuous map, associated operators: $\hat{T} : C(X) \to C(X)$ pull back of functions and $T_* : \mathcal{M} \to \mathcal{M}$ push forward of measures. T_* is continuous in the weak-$*$ topology. $\mathcal{M}_{\text{inv}}$ as the fixed point set of T_*, closed and convex.

Krylov-Bogolyubov theorem, proof based on the ergodic averages of Dirac measures, discussion, alternative proof based on the Schauder fixed point theorem.

Example: $T : [0,1] \to [0,1]$, $Tx = x/2$, $\mathcal{M}_{\text{inv}} = \{\delta_0\}$. Outlook: $\mathcal{M}_{\text{inv}}$ for irrational rotations (unique ergodicity) and the doubling map ($\mathcal{M}_{\text{inv}}$ is large).

Extreme points of a convex set. $\mu \in \mathcal{M}_{\text{erg}}$ if and only if it is an extreme point of $\mathcal{M}_{\text{inv}}$. As a byproduct: if $\mu \in \mathcal{M}_{\text{erg}}$, $\mu_1 \in \mathcal{M}_{\text{inv}}$, $\mu_1 \ll \mu$, then $\mu_1 = \mu$.

March 29

If $m, \mu \in \mathcal{M}_{\text{erg}}$, either $m = \mu$ or $m \perp \mu$.

von Neumann's ergodic theorem (with proof). Birkhoff's ergodic theorem (without proof).

Equivalent characetrizations of ergodicty by the averaged correlations of sets and functions.

Definition of mixing via sets and functions, analogy: decay of correlations. Mixing implies ergodicity.

April 4

Irrational rotations: ergodicity, unique ergodicity, lack of mixing. Weyl's theorem. Arnold's problem on the frequency of the first decimal digits in the sequence 2^n.

Reminder: one sided and double sided full shifts, cylinder sets.

Bernoulli shift.

April 5

Mixing of Bernoulli shifts.

Recap of finite Markov chains. Transition probabilities, stochastic matrices, adjacency matrix. Irreducible and primitive (irreducible aperiodic) cases. Stationary distribution. Perron's theorem, spectral gap, exponential convergence to the stationary distribution (in the primitive case).

April 11

Adjacency matrix. Topological Markov chains: phase space as a compact invariant subspace of the full shift. Outlook: subshifts of finite type.

Markov shifts. Measure of cylinder sets via the stationary distribution and the transition probabilities. For primitive transition matrix the Markov shift is mixing.

April 12

Banach space of Hölder continuous functions. Definition of Exponential decay of correlations. Conditional expectations w. r. to the σ-algebra generated by cylinder sets of length ℓ. Exponential decay of correlations for mixing Markov shifts.

ω-limit sets, periodic, recurrent and non-wandering points.