Markov Chains and Dynamical Systems, Spring 2025

Homework problem set #2 . Due on March 13, Thursday

Problems 1.-3. are from Durrett, R.: Essentials of Stochastic Processes (Section 1.12), available at the author's webpage.

1. Problem 1.37
2. Problem 1.46
3. Problem 1.54 (Note: a reversible stationary distribution is the same as a stationary distribution with detailed balance. See section 1.6.3 for this terminology.)
4. Consider a „tiny" chessboard made of a 5 by 5 grid of squares. A knight moves randomly on this chessboard by choosing evenly out of the possible steps that it can make.
   (a) Is the associated Markov chain irreducible? Is it aperiodic?
   (b) Compute the stationary distribution.
   (c) Starting from the center of the board, what is the average number of steps the knight takes before getting back to this central position?
   (d) Again starting from the center, what is the chance the knight is back to its starting position after 1000 steps? And after 999 steps?
5. John has liability insurance for his car. The insurance company puts drivers into 4 categories: $1, 2, 3, 4$. If a driver does not cause any accidents for an entire year, he moves up by 1 category (if he was in category 4, he stays there). If a driver causes a major accident, next year he goes into category 1. If a driver causes a minor accident, but no major accidents during a year, next year he moves down by 1 category (if he was in category 1, he stays there). John causes a major accident during a year with probability $1/12$, and the probability that he causes a minor accident but no major accidents during a year is $1/4$.
   (a) Model this process with a Markov chain. What are the states? Calculate the transition matrix. Is the Markov chain irreducible? Is it aperiodic?
   (b) What is the conditional probability that John will be in category 2 two years from now, assuming that now he is in category 4?
   (c) What is the probability that he will be in category 2 ten years from now?
   (d) In the long run, how often does he move from category 3 to category 4 on average?
   (e) For each category, the annual cost is respectively $120000, 72000, 54000, 36000$ HUF. What is the long-term average annual cost paid by John?