Markov Chains and Dynamical Systems, Spring 2025

Homework problem set #4 . Due on April 24, Thursday

1. Consider the one dimensional map $T_\lambda : \mathbb{R} \to \mathbb{R}$, $T_\lambda x = x^3 - \lambda x$, where the parameter λ satisfies $-\infty < \lambda \leq 1$. Investigate the λ-dependence of

   (a) the fixed points and their stability properties.

   (b) the asymptotic behavior of the orbit $T_\lambda^n x$, $n \geq 0$ for any initial condition $x \in \mathbb{R}$.

2. (a) Prove that $\log_2(3)$ is an irrational number.

   (b) For some $m \geq 2$, let $w = w_1 w_2 \dots w_m = 1 w_2 \dots w_m$ be a binary word of length m that starts with 1 but otherwise arbitrary, that is, $w_1 = 1$ while $w_i \in \{0, 1\}$, $i = 2, \dots m$. Show that there exists some $n \geq 1$ such that the first m digits in the binary code of 3^n coincide with w.

   (c) Consider the first two digits in the binary code for the sequence of numbers $3, 9, 27, \dots 3^n, \dots$. There are two options: 10 and 11. Which one is more frequent? (Compute the asymptotic frequencies of these two options.)

3. Recall that, given two dynamical systems $T_1 : M_1 \to M_1$ and $T_2 : M_2 \to M_2$,

   - T_2 is a *factor* of T_1 if there is a map $\Phi : M_1 \to M_2$ such that $\Phi \circ T_1 = T_2 \circ \Phi$. In such a case, Φ is called a *semiconjugacy*.
   - If, moreover, Φ is a bijection (one-to-one and onto) then it is called a *conjugacy* (and the two dynamical systems are *conjugate*).

   For any $\alpha \in (0, 1)$, let $R_\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ denote the rotation of the circle by α. Given α, let $\beta = 2\alpha \,(mod\, 1)$ and $\gamma = 1 - \alpha$.

   (a) Show that R_β is a factor of R_α. (Moreover, the semiconjugacy is continuous).

   (b) Show that R_γ and R_α are conjugate. (Moreover, the conjugacy is a homeomorphism of $\mathbb{S}^1$, that is, a continuous bijection with a continuous inverse).

4. Consider the doubling map $T : \mathbb{S}^1 \to \mathbb{S}^1$, $Tx = 2x \pmod 1$. Let $D \subset \mathbb{S}^1$ denote the set of points x such that $\{T^n x | n = 0, 1, 2 \dots\}$ is dense in $\mathbb{S}^1$. Prove that D is uncountable.

   Bonus problem: Prove that $\lambda(D) = 1$ (where λ is the Lebesgue measure).

5. Consider the logistic map $T_a x = ax(1 - x)$ with $a > 2 + \sqrt{5}$. Show that there exists some $\lambda > 1$ such that $|T_a'(x)| > \lambda$ whenever $x \in I_0 \cup I_1$. (Recall that we denoted $T_a^{-1}[0, 1] = I_0 \cup I_1$, where I_0 and I_1 are disjoint intervals.)