# Markov Chains and Dynamical Systems, Spring 2025

## Homework problem set #5 . Due on May 9, Friday

1. Consider $T : [0,1] \to [0,1]$, $Tx = 4x(1-x)$ (the logistic map with $a = 4$) . Verify that $T$ has an absolutely continuous invariant (probability) measure, with density $\rho(x) = C(x(1-x))^{-1/2}$. ($C =?$)

2. Let $T : \mathbb{T}^2 \to \mathbb{T}^2$ be a hyperbolic toral automorphism. Show that, for the matrix $A \in SL(2,\mathbb{Z})$ associated to $T$, the eigenvalues are irrational numbers while the eigendirections, as lines on $\mathbb{R}^2$, have irrational slope.

3. Let $T : \mathbb{T}^2 \to \mathbb{T}^2$ be a hyperbolic toral automorphism. Show that $\underline{x} \in \mathbb{T}^2$ is a periodic point for $T$ if and only if both of its coordinates are rational.

4. For some $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$, that is, a $2 \times 2$ matrix with *real* entries $a, b, c, d$ and determinant 1, consider the *one dimensional* map $T_A : \mathbb{R} \setminus \left\{ -\frac{d}{c} \right\} \to \mathbb{R}$, $T_A(x) = \frac{ax+b}{cx+d}$. Such maps are called *fractional linear transformations.*

   (a) Show that, for two such matrices, $A_1, A_2 \in SL(2,\mathbb{R})$, $A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$, $i = 1, 2$, we have $T_{A_2} \circ T_{A_1} = T_{A_2 \cdot A_1}$, where $\cdot$ is matrix multiplication. In particular, for $A \in SL(2,\mathbb{R})$, we have $(T_A)^n = T_{A^n}$, $n \in \mathbb{Z}$.

   (b) Check how many fixed points such a fractional linear transformation $T_A$ can have. In particular, try to find conditions in terms of $Tr\,A$ that determine the number of fixed points. (Keep in mind that $det\,A = 1$.)

   *Comment:* Literally, fractional linear transformations cannot be regarded as dynamical systems acting on the phase space $\mathbb{R}$, as there is a singularity at $-\frac{d}{c}$. However, they can be regarded as acting on the phase space $M = \mathbb{R} \cup \{\infty\}$, where $\infty$ is obtained by identifying $+\infty$ and $-\infty$. With this notation, $T\left(-\frac{d}{c}\right) = \infty$ while $T(\infty) = \frac{a}{c}$.

5. Continuation of the previous problem. For the matrices $A$ below, sketch the graph of $T_A$. Find if there are any fixed points, and if you find some, check their stability properties (attracting or repelling). Furthermore, check if there are any period 2 or period 3 orbits.

$$(a)\ A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \qquad (b)\ A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}; \qquad (c)\ A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

   *Comment:* Some of these examples seem to violate Sharkovsky's theorem. Note, however, that fractional linear maps act on $\mathbb{R} \cup \{\infty\}$ instead of $\mathbb{R}$, see the comment for the previous problem.