Markov Chains and Dynamical Systems, Spring 2025

Sample problems on Dynamical Systems for the Final Exam

The Final Exam is cumulative.
There will be 1 or 2 problems on Markov chains,
and 3 or 4 problems on Dynamical Systems.

Other types of problems may also occur on the exam.
Anything we had in class, quizzes or homeworks can be relevant.

1. (a) Find the binary code of $x = \frac{5}{9}$, that is, the sequence of digits $x_k \in \{0,1\}, \quad k = 1,2,\dots$ such that $\frac{5}{9} = \sum_{k=1}^{\infty} x_k 2^{-k}$. (*Hint:* Consider the orbit of x under the doubling map.)

   (b) The rational number $y \in (0,1)$ has binary code 110111011101... (the word 1101 is repeated periodically). Find p, q coprime integers such that $y = p/q$.

   (c) Recall that the orbit of a point $z \in (0,1)$ equidistributes under the dobling map if for any interval $I \subset (0,1)$ we have

$$\lim_{n\to\infty} \frac{\#\{k = 0,\dots,n-1 \mid T^k z \in I\}}{n} = |I|$$

   where $|I|$ is the length (ie. the Lebesgue measure) of I. Find the binary code for some $z \in I$ such that the orbit of z (under the doubling map) is dense on $\mathbb{S}^1$, but does not equidistribute.

2. (a) Prove that $\log_5(3)$ is an irrational number.

   (b) Consider the first symbols in the base-5 expansions for the sequence of numbers $3, 9, 27, \dots 3^n, \dots$. Compute the asymptotic frequencies of the four possible options: $1, 2, 3$ and 4.

3. Consider the one dimensional map $T : \mathbb{R} \to \mathbb{R}$, $Tx = x + \sin x$.

   (a) Find all fixed points and discuss their stability properties.

   (b) Describe the asymptotic behavior of the orbit $T^n x$, $n \geq 0$ for any initial condition $x \in \mathbb{R}$.

4. Consider the one parameter family of maps $T_\lambda : \mathbb{R} \to \mathbb{R}$, $T_\lambda x = \lambda - x^2$, with $\lambda \in \mathbb{R}$.

   (a) For each $\lambda \in \mathbb{R}$, find all fixed points and discuss their stability properties.

   (b) What does it mean that for some $\lambda_1 \in \mathbb{R}$ the family has a saddle-node bifurcation? Does there exist such a λ_1 for this particular family? Why?

   (c) What does it mean that for some $\lambda_2 \in \mathbb{R}$ the family has a period doubling bifurcation? Does there exist such a λ_2 for this particular family? Why?

5. Consider the map $T : \mathbb{R} \to \mathbb{R}$,

$$T(x) = \begin{cases} 10x & \text{if } x \leq \frac{1}{2} \\ 10 - 10x & \text{if } x > \frac{1}{2}. \end{cases}$$

   For $x_0 \in \mathbb{R}$, let $x_n = T^n x_0$ for $n \geq 1$.

(a) Show that $\lim_{n\to\infty} x_n = -\infty$ whenever $x_0 \notin [0,1]$.

(b) Let $\Lambda_1 = \{x \in [0,1) \,|\, Tx \in [0,1)\}$. Show that Λ_1 consists of two intervals. How can you characterize the numbers $x \in \Lambda_1$ by their decimal digits?

(c) For any integer $k \geq 2$, let $\Lambda_k = \{x \in [0,1) \,|\, T^j x \in [0,1); j = 1, \ldots k\}$. Show that Λ_k consists of 2^k intervals. How can you characterize the numbers $x \in \Lambda_k$ by their decimal digits?

(d) $\Lambda = \{x \in [0,1) \,|\, T^j x \in [0,1); \forall j \geq 1\}$. How can you characterize the numbers $x \in \Lambda$ by their decimal digits?

6. Consider the map $T : [0,1] \to [0,1]$,

$$T(x) = \begin{cases} \frac{3}{2}x & \text{if } 0 \leq x < \frac{2}{3}, \\ 2x - \frac{4}{3} & \text{if } \frac{2}{3} \leq x \leq 1. \end{cases}$$

Is Lebesgue measure invariant for T? If yes, explain why, if no, find another absolutely continuous invariant measure (ie. an invariant density).

7. Consider the linear maps $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x) = Ax$ for the matrices A below. In each case, describe the asymptotic behavior and sketch the phase portrait. In hyperbolic cases, determine the stable and unstable subspaces (W^s and W^u).

(a) $\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 1 \\ 0 & 1/3 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1/5 \\ 1/5 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}$.

8. Consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

(a) Determine the eigenvalues and the eigenvectors of A.

(b) Let $T : \mathbb{T}^2 \to \mathbb{T}^2$ denote the toral automorphism associated to A, that is, the CAT map. Find some $\underline{x} \in \mathbb{T}^2$ ($\underline{x} \neq (0,0)$) that is homoclinic to the fixed point $(0,0)$.

9. (Related to topics to be discussed in class on May 8.) Let $\Sigma = \{0,1\}^{\mathbb{Z}}$ (the space of bi-infinite binary sequences), and let $\sigma : \Sigma \to \Sigma$ denote the corresponding two-sided full shift.

(a) How do you define the distance of $d(\underline{i}, \underline{j})$ for two points $\underline{i}, \underline{j} \in \Sigma$? (Let $\underline{i} = \ldots i_{-1} i_0 i_1 \ldots$ and similarly for $\underline{j}$).

(b) Find all the fixed points of σ.

(c) Find some $\underline{i} \in \Sigma$ which is periodic for σ with prime period 7.

(d) Find some $\underline{j} \in \Sigma$ that satisfies both of the following two criteria. (i) $d(\underline{i}, \underline{j}) < 0.01$ (with $\underline{i}$ from problem (c)). (ii) $\underline{j}$ is homoclinic to (one of) the fixed point(s) from problem (b).