Markov Chains and Dynamical Systems, Fall 2026

Log: a brief summary of the classes

September 8

Introduction:

Dynamical systems: phase space M , map $T : M \to M$. Orbit, periodicity, asymptotic behavior. Simple examples on $M = \mathbb{R}$ or $M = [0, 1)$ with regular vs. chaotic patterns. Billiards.

Markov chains: state space, transition probabilities. A simple example: the weather chain. Stationary distribution and its relation to the following questions: on the average, what is the long term proportion of sunny days? If today it is sunny, what is the expected number of days that it is sunny again? Formula for the stationary distribution for chains with two states. *Random walks* on $\mathbb{Z}$ and $\mathbb{Z}^2$

Relation of the two topics: evolution at different time scales.

September 10

Probability recap: $(\Omega, \mathcal{F}, \mathbb{P})$ - sample spaces, σ-algebra of events, axioms of probability. Examples: roll two dice, infinite sequences of coin tosses. Inclusion-exclusion formula.

Conditional probability. Multiplication rule. Independence.

Discrete random variables: mass function, expected value $\mathbb{E}X$. The binomial and the geometric distributions. Methods for computing $\mathbb{E}X$ – sums of indicators, and $\sum\limits_{k\geq 1} \mathbb{P}(X \geq k)$.

September 15

Pairwise and complete independence of events $A_1, A_2, ...$

Independent random variables, i.i.d. sequences.

Definition of a (homogeneous)*Markov chain.* Transition matrix and its properties.

Examples: Gambler's ruin, Ehrenfest chain, Weather chain, Social mobility chain. Description of the transition graphs and the matrices $p(i, j)$ for these examples.

Absorbing sates. First return time T_i for $i \in S$, the return probabilities $\rho_{ii} = \mathbb{P}(T_i < \infty \,|\, X_0 = i)$. Transient ($\rho_{ii} < 1$) and recurrent ($\rho_{ii} = 1$) states. Discussion of the above examples from this perspective.

September 17

Recap of setting: state space S, Markov chain, transition probabilities. Stochastic matrices ($\sum_j p(i, j) = 1\ \forall i$). Weather chain, Social mobility chain.

Multistep transition probabilities. Chapman-Kolmogorov equations. $p_n(i, j) = (p^n)_{ij}$. Inspection of p^n for the above examples.

Evolution of distributions. Stationary distribution $\pi(i)$ as a (row) eigenvector corresponding to the eigenvalue 1 for multiplication by $p(i, j)$ from the right.

September 22

Computing π for chains with two states.

Computing π for chains with more than two states. Question of uniqueness.

Recap of notations: $P_x(\cdot) = \mathbb{P}(\cdot | X_0 = x)$, $T_x = \min\{n \geq 1 | X_n = x\}$ (first hitting/return time), $\rho_{xy} = P_x(T_y < \infty)$.
Transience ($\rho_{xx} < 1$) – recurrence ($\rho_{xx} = 1$) dichotomy.

Relation $x \to y$ and equivalence relation $x \leftrightarrow y$. Closed and irreducible classes.
Proposition: if C is a finite, closed and irreducible class, then all states $x \in C$ are recurrent.
Corollary – classification of states: $S = T \cup C_1 \cup \dots C_K$, where T is the collection of transient states, while C_i are closed irreducible classes.