MARKOV CHAINS AND DYNAMICAL SYSTEMS

Péter Bálint
E-mail: `balint.peter@ttk.bme.hu`

Course Description: The course aims to give an introduction to two challenging mathematical disciplines, dynamical systems and Markov chains. In rough terms, the main common theme is to provide mathematical tools to describe phenomena evolving in time. While dynamical systems and Markov chains share several common features, they are also somewhat complementary in the sense that they study time evolution from the deterministic and the stochastic perspectives, respectively. Of course, there is a lot more to say about the mathematical aspects of time evolution. Yet, the present course focuses on simple examples – dynamical systems in one or two dimensions, and Markov chains with discrete state spaces – which can be studied by elementary tools. This way the students can learn about the main conceptual aspects and the key phenomena without going too deep into the technical complications. It is expected that the experience gained at studying these simple examples will be utmost useful at later stages of their curriculum. Dynamical systems and Markov chains are essential in several major areas of pure and applied mathematics, extensively used for example in financial mathematics or internet search engines, and occur as models for a wide range of engineering, economical, physical, biological and sociological phenomena. Throughout the course, an emphasis is put on highlighting such connections.

Course plan: The prerequisites for this course do not go beyond calculus (including standard properties of numerical sequences and series) and a first course in linear algebra (operations on vectors and matrices, eigenvalues and eigenvectors, etc.). Some familiarity with further topics in analysis and probability is useful, but this will be treated in a self-contained manner. In the first half of the course Markov chains, in the second half dynamical systems are considered, a roughly equal amount of time is spent on these two topics. Throughout, we focus on investigating specific examples. In more detail:

- *Markov chains*
  - The concept of a Markov chain (finite state space, discrete time). The transition matrix. Multistep transition probabilities.
  - Classification of states. Transience and recurrence. Irreducible classes.
  - Stationary distributions. Irreducibility and (a)periodicity. Limit behavior.
  - Detailed balance. Birth and death chains.
  - Gambler's ruin. Exit distributions and exit times.
  - A glimpse at infinite state space. Random walks. Branching processes.
- *Dynamical systems*
  - Phase spaces and maps. Two examples – rotations of the circle and the doubling map – to demonstrate regular and chaotic behavior. Periodic points. Density and equidistribution of orbits. Dyadic representation as symbolic coding. Arnold's problem on the frequency of the first decimal digits of 2^n.
  - One dimensional maps. Cobweb plots. Attracting and repelling fixed points. The logistic family $Tx = \mu x(1-x)$. Period doubling bifurcations. Emergence of a Cantor set as a repeller for $\mu > 4$. "Period three implies chaos."

- Linear maps of the plane. Phase portraits. Smale's horseshoe, Cantor sets. The CAT map. Hyperbolic dynamics and the shadowing property.

Textbooks: The two main topics of the course, dynamical systems and Markov chains, are typically treated separately in the literature. Thus there is no recommendation for a single book that covers the material discussed. Nevertheless, there are several excellent textbooks that can be used (and will be referred to) in course of the semester. In particular, the following books/lecture notes are recommended.

- Durrett, R.: Essentials of Stochastic Processes, available at the author's webpage.
- Devaney, R.: An Introduction to Chaotic Dynamical Systems
- Chernov, N.: Dynamical systems class notes, available at the author's webpage.

Office hours: By appointment.

Grading policy:

- *Quizzes*, on a weekly basis, consisting of simple questions related to the recently studied material. If the student follows classes, these questions can be answered essentially immediately, or after a couple of minutes thinking. Contribution to the total score 20%.
- *Homework problem sets.* At most six homework problem sets will be assigned in course of the semester. Contribution to the total score 20%.
- *Midterm exam* approximately on the eighth week of the semester, focusing mostly on Makov chains. The allocated time is 60 minutes. Contribution to the total score 25%.
- *Final exam* is cumulative, but more emphasis is put on dynamical systems. The allocated time is 100 minutes. Contribution to the total score 35%.

Letter grades will be based on the total score T, approximating the following standards:

$$
\begin{array}{rrr}
97\% \le T : A^+ & 87\% \le T < 97\% : A & 85\% \le T < 87\% : A^- \\
82\% \le T < 85\% : B^+ & 72\% \le T < 82\% : B & 70\% \le T < 72\% : B^- \\
67\% \le T < 70\% : C^+ & 57\% \le T < 67\% : C & 55\% \le T < 57\% : C^- \\
 & 40\% \le T < 55\% : D & \\
 & T < 40\% : F &
\end{array}
$$