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The algebra of the canonical commutation relation

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A teljes könyv címe :

An invitation to the algebra of the canonical commutation relation
(Leuven University Press, 1990.)

Már elfogyott, de megtalálható:

http://www.renyi.hu/˜ petz/pdf/ccr.pdf

Chapter 1

The operators P and Q

1.1 The Hilbert space

The Hilbert space in this chapter is $L^2(\mathbb{R})$ with complex-valued functions. The well-known formulas are

$$L^2(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} \, : \, \int |f(x)|^2 \, dx < \infty \right\},$$

$$\langle f, g \rangle = \int \overline{f(x)} g(x) \, dx, \qquad \|f\| = \left[\int |f(x)|^2 \, dx \right]^{1/2}.$$

It is important to have an orthonormal basis in the Hilbert space $L^2(\mathbb{R})$. We shall use the Hermite functions which are based on the Hermite polynomials. The **Hermite polynomials**

$$H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \qquad (n = 0, 1, \ldots) \tag{1.1}$$

are orthogonal in the Hilbert space $L^2(\mathbb{R}, e^{-x^2} \, dx)$, they satisfy the recursion

$$H_{n+1}(x) - 2x H_n(x) + 2n H_{n-1}(x) = 0 \tag{1.2}$$

and the differential equation

$$H_n''(x) - 2x H_n'(x) + 2n H_n(x) = 0. \tag{1.3}$$

An important useful formula is

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = e^{2xt - t^2}. \tag{1.4}$$

The normalized Hermite polynomials

$$\tilde{H}_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) \tag{1.5}$$

form an orthonormal basis. From this basis of $L^2(\mathbb{R}, e^{-x^2} \, dx)$, we can get easily a basis in $L^2(\mathbb{R})$:

$$\varphi_n(x) := e^{-x^2/2} \tilde{H}_n(x) \qquad (n = 0, 1, \ldots). \tag{1.6}$$

These are called **Hermite functions**. In terms of the Hermite functions equation (1.2) becomes

$$x\varphi_n(x) = \frac{\sqrt{n}\varphi_{n-1}(x) + \sqrt{n+1}\varphi_{n+1}(x)}{\sqrt{2}}. \tag{1.7}$$

So the operator $Q : \varphi_n(x) \mapsto x\varphi_n(x)$ has a matrix

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & \sqrt{2} & 0 & \dots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \dots \\ \dots & & & & \end{bmatrix}.$$

The **Fourier transform** of a function $f \in L^1(\mathbb{R})$ is defined as follows.

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int e^{-\mathrm{i}tx} f(x)\, dx. \tag{1.8}$$

If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then

$$\|f\|^2 = \|\hat{f}\|^2$$

and the Fourier transform extends to a unitary $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$. If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then

$$(\mathcal{F}^{-1} f)(t) = \frac{1}{\sqrt{2\pi}} \int e^{\mathrm{i}tx} f(x)\, dx. \tag{1.9}$$

If $x^n f(x)$ is integrable for every $n \in \mathbb{N}$, then the Fourier transform is infinetely many times defferentiable. It is a very useful set of functions is the **Schwarz space**:

$$\mathcal{S}(\mathbb{R}) = \{f : \lim_{x\to\pm\infty} x^m f^{(n)}(x) = 0 \text{ for every } n, m \in \mathbb{N}\}.$$

The Hermite functions are in the Schwarz space, it follows that the Schwarz space is dense in $L^2(\mathbb{R})$. It is true that $\mathcal{F}(\mathcal{S}(\mathbb{R})) \subset \mathcal{S}(\mathbb{R})$.

1.2 Some unbounded operators

Let $Q : L^2(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$ be the multiplication by the variable: $\mathcal{D}(Q) = \{f \in L^2(\mathbb{R}) : xf(x) \in L^2(\mathbb{R})\}$ and for $f \in \mathcal{D}(Q)$ we have $(Qf)(x) = xf(x)$. It follows from the definition that the unbounded operator Q is symmetric, $Q \subset Q^*$, moreover

$$Q(\mathcal{S}(\mathbb{R})) \subset \mathcal{S}(\mathbb{R})$$

holds. We show that Q is self-adjoint.

Lemma 1.1 *The operator Q is self-adjoint.*

Proof. If $g \in \mathcal{D}(Q^*)$ and $Q^*g = g^*$, then $\langle g, Qf\rangle = \langle g^*, f\rangle$ for every $f \in \mathcal{D}(Q)$. In other words

$$\int \left[\overline{g}(x)x - \overline{g^*}(x)\right] f(x)\, dx = 0\,.$$

In place of f let us put the function $1_{[-n,n]}\,[g(x)x - g^*(x)]$ which is square integrable and has compact support. Therefore, it is in $\mathcal{D}(Q)$. So

$$g(x)x - g^*(x) = 0$$

almost everywhere in the interval $[-n, n]$, it follows that in $\mathbb{R}$ as well. We obtain that $g \in \mathcal{D}(Q)$ and $Qg = g^*$. We find that $Q^* \subset Q$ and we conclude that Q is self-adjoint. $\square$

If the operators a and a^+ are defined as

$$a\varphi_n = \sqrt{n}\varphi_{n-1}, \qquad a^+\varphi_n = \sqrt{n+1}\varphi_{n+1} \tag{1.10}$$

with $a\varphi_0 = 0$, then (1.7) is

$$Q = \frac{1}{\sqrt{2}}(a + a^+). \tag{1.11}$$

Example 1.2 Q is a self-adjoint operator, therefore it has a spectral decomposition

$$Q = \int \lambda\, dE(\lambda),$$

where the spectral measure of a subset $H \subset \mathbb{R}$ is defined as

$$E(H)f = 1_H f.$$

It follows that

$$e^{\mathrm{i}tQ}f(x) = e^{\mathrm{i}tx}f(x). \tag{1.12}$$

For $t \in \mathbb{R}$ this is a one-parameter group of unitaries. $\square$

Example 1.3 Let $C_0^\infty(\mathbb{R})$ be the space of infinitely many times differentiable functions of compact support and $Q_0 := Q|C_0^\infty(\mathbb{R})$. We show that the closure of Q_0 is Q.

We have to show that for $f \in \mathcal{D}(Q)$ there is a sequence $g_n \in C_0^\infty$ such that $g_n \to f$ and $g_n(x)x \to xf(x)$ in L^2-norm. Let $f_n := 1_{[-n,n]}f$ and choose $g_n \in C_0^\infty$ such that its support is in $[-n, n]$ and $\|f_n - g_n\| \le n^{-2}$. Then $g_n \to f$ and

$$\int_{-n}^{n} |xg_n(x) - xf_n(x)|^2\, dx \le n^2 \int_{-n}^{n} |g_n(x) - f_n(x)|^2\, dx \le n^{-2}.$$

This shows that the limit of the sequences $xg_n(x)$ and $xf_n(x)$ are the same. The latter converges to $xf(x)$, and so does the first one. $\square$

Let

$$(Pf)(x) = \frac{1}{\mathrm{i}}f'(x) \qquad (f \text{ is differentiable and } f, f' \in L^2(\mathbb{R})).$$

The Schwarz set is in the domain and $P(\mathcal{S}(\mathbb{R})) \subset \mathcal{S}(\mathbb{R})$ holds. From the equation

$$\frac{\partial}{\partial x}\Big(H_n(x)e^{-x^2/2}\Big) = H_n'(x)e^{-x^2/2} - xH_n(x)e^{-x^2/2},$$

one can obtain

$$P\varphi_n := \frac{1}{\mathrm{i}}\varphi_n' = \frac{\sqrt{n}\varphi_{n-1} - \sqrt{n+1}\varphi_{n+1}}{\mathrm{i}\sqrt{2}}, \tag{1.13}$$

that is

$$P = \frac{\mathrm{i}}{\sqrt{2}}(a^+ - a). \tag{1.14}$$

Therefore,

$$a = \frac{1}{\sqrt{2}}(Q + \mathrm{i}P), \qquad a^+ = \frac{1}{\sqrt{2}}(Q - \mathrm{i}P).$$

In the basis of Hermite functions P has the following matrix:

$$P = \frac{-i}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ -1 & 0 & \sqrt{2} & 0 & \dots \\ 0 & -\sqrt{2} & 0 & \sqrt{3} & \dots \\ \dots & & & & \end{bmatrix}.$$

The relation of P and Q can be deduced from the formula

$$\frac{\partial}{\partial t}(\mathcal{F}^{-1}f)(t) = \frac{1}{\sqrt{2\pi}}\int e^{\mathrm{i}tx}\mathrm{i}xf(x)\,dx.$$

We obtain $P\mathcal{F}^{-1}f = \mathcal{F}^{-1}Qf$ or

$$Pf = \mathcal{F}^{-1}Q\mathcal{F}f \qquad (f \in \mathcal{S}(\mathbb{R})). \tag{1.15}$$

From this relation we can get

$$(e^{\mathrm{i}tP}f)(x) = f(x+t) \qquad (t \in \mathbb{R}). \tag{1.16}$$

The Schwarz space is in the domain of the linear combinations and of the product of P and Q. From the definition the canonical commutation relation

$$(QP - PQ)f = \mathrm{i}f \qquad (f \in \mathcal{S}(\mathbb{R})) \tag{1.17}$$

follows.

1.3 Unitaries

The unitary operators $e^{\mathrm{i}tQ}$ and $e^{\mathrm{i}uP}$ already appeared, the formulas (1.12) and (1.16) are simple, but formulation in the orthonormal basis is more complicated. Next the two unitaries will be combined.

Lemma 1.4 *For every real t amd u*

$$e^{\mathrm{i}(tQ+uP)} = \exp(\mathrm{i}tu/2)e^{\mathrm{i}tQ}e^{\mathrm{i}uP} = \exp(-\mathrm{i}tu/2)e^{\mathrm{i}uP}e^{\mathrm{i}tQ}$$

holds.

Proof. The usual reference is to the Baker-Campbell-Hausdorff formula which says that

$$e^A e^B = e^{A+B} e^{(AB-BA)/2}$$

if A and B commute with $[A, B] := AB - BA$. □

For $z = x + \mathrm{i}y \in \mathbb{C}$ set

$$W(z) = \exp \mathrm{i}\sqrt{2}(xP + yQ) = \exp \mathrm{i}(za^+ + \bar{z}a). \tag{1.18}$$

The previous lemma gives the following relation

$$W(z)W(z') = W(z + z') \exp(\mathrm{i}\,\mathrm{Im}\,(\bar{z}z')) \qquad (z, z' \in \mathbb{C}). \tag{1.19}$$

This shows that the (finite) linear combination of the unitaries $W(z)$ is a *-algebra. If we take the closure in $B(L^2(\mathbb{R}))$, then we get a C*-algebra $\mathcal{A}$, in the next chapter this will be generalized.

The proof of the next statement is in [17]. We can see that the unbounded linear operators $aP + bQ$ are rather simple in the orthonormal basis $\{\varphi_n : n = 0, 1, \ldots\}$, but the action of the unitary $W(x + \mathrm{i}y)$ is much more technical.

Lemma 1.5

$$\langle \varphi_n, W(z)\varphi_n \rangle = e^{-|z|^2/2} L_n(|z|^2)$$

holds, where

$$L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k n!}{k!\,(n-k)!\,k!} x^k\,.$$

A particular case is

$$\langle \varphi_0, W(z)\varphi_0 \rangle = e^{-|z|^2/2}. \tag{1.20}$$

Since we have

$$\sum_{n=0}^{\infty} t^n L_n(x) = \frac{1}{1-t} \exp\Big(-\frac{xt}{1-t}\Big) \tag{1.21}$$

for $|t| < 1$ and $x \in \mathbb{R}^+$, it is obtained that

$$\sum_{n=0}^{\infty} \mu^n (1-\mu) \langle \varphi_n, W(z)\varphi_n \rangle = \exp\Big(-\frac{|z|^2}{2}\frac{1+\mu}{1-\mu}\Big)$$

for $0 \le \mu < 1$, or

$$\sum_{n=0}^{\infty} \frac{1}{1+\lambda}\left(\frac{\lambda}{1+\lambda}\right)^n \langle \varphi_n, W(z)\varphi_n \rangle = \exp\Big(-\frac{|z|^2}{2} - \lambda|z|^2\Big) \tag{1.22}$$

for $\lambda \ge 0$. Note that

$$D = \sum_{n=0}^{\infty} \frac{1}{1+\lambda}\left(\frac{\lambda}{1+\lambda}\right)^n |\varphi_n\rangle\langle\varphi_n| \tag{1.23}$$

is a statistical operator, since the eigenvalues are positive and the sum is 1. (1.22) is rewritten in the form

$$\mathrm{Tr}\, DW(z) = \exp\Big(-\frac{|z|^2}{2} - \lambda|z|^2\Big).$$

1.4 Exercises

1. Check that in formula (1.10) $a^* = a^+$ holds.
2. Compute the entropy of the density (1.23).
3. Check that
$$a = \frac{Q + \mathrm{i}P}{\sqrt{2}}.$$

Chapter 2

The C*-algebra of the canonical commutation relation

If $\mathcal{H}$ is a complex Hilbert space then $\sigma(f,g) = \mathrm{Im}\,\langle f, g\rangle$ is a nondegenerate symplectic form on the real linear space $\mathcal{H}$. (Symplectic form means $\sigma(x,y) = -\sigma(y,x)$.) (H,σ) will be a typical notation for a Hilbert space and it will be called symplectic space.

Let (H,σ) be a symplectic space. The C*-algebra of the **canonical commutation relation** over (H,σ), written as CCR(H,σ), is by definition a C*-algebra generated by elements $\{W(f) : f \in H\}$ such that

(i) $W(-f) = W(f)^* \qquad (f \in H)$,

(ii) $W(f)W(g) = \exp(\mathrm{i}\,\sigma(f,g))\,W(f+g) \qquad (f,g \in H)$.

Condition (ii) tells us that $W(f)W(0) = W(0)W(f) = W(f)$. Hence $W(0)$ is the unit of the algebra and it follows that $W(f)$ is a unitary for every $f \in H$.

The typical example comes from Lemma 1.4, see (1.18) and (1.19).

Example 2.1 If $f, g \in \mathcal{H}$ are orthogonal, then

$$W(f)W(g) = W(f+g) = W(g+f) = W(g)W(f).$$

It follows that for $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, CCR$(\mathcal{H})$ is isomorphic to CCR$(\mathcal{H}_1) \otimes$ CCR$(\mathcal{H}_2)$. $\square$

Theorem 2.2 *For any symplectic space (H,σ) the C*-algebra CCR(H,σ) exists and it is unique up to isomorphism.*

Proof. To establish the existence will be easier than proof of the uniqueness. Consider H as a discrete abelian group (with the vectorspace addition).

$$l^2(H) = \left\{ F : H \to \mathbb{C} : \sum_{x \in H} |F(x)|^2 < +\infty \right\}$$

is a Hilbert space. (Any element of $l^2(H)$ is a function with countable support.) Setting

$$(R(x)F)(y) = \exp(i\,\sigma(y,x))F(x+y) \qquad (x, y \in H) \tag{2.1}$$

we get a unitary $R(x)$ on $l^2(H)$ and one may check that

$$R(x_1)R(x_2) = \exp(i\,\sigma(x_1,x_2)R(x_1+x_2).$$

The norm closure of the set

$$\left\{\sum_{i=1}^{n} \lambda_i\, R(x_i) \quad : \quad \lambda_i \in \mathbb{C}\,,\;\; 1 \le i \le n\,,\;\; n \in \mathbb{N}\,,\;\; x_i \in H\right\}$$

in $B(l^2(H))$ is a C*–algebra fulfulling the requirements (i) and (ii). Let us denote this C*-algebra by $\mathcal{A}$.

Assume that $\mathcal{B} \subset B(\mathcal{H})$ is another C*-algebra generated by elements $W(x)$ $(x \in H)$ satisfying (i) and (ii). We have to show an isomorphism $\alpha : \mathcal{A} \to \mathcal{B}$ such that $\alpha(R(x)) = W(x)$ $(x \in H)$. α will be constructed in several steps.

We shall need the Hilbert space

$$l^2(H,\mathcal{H}) = \left\{A \,:\, H \to \mathcal{H} \,:\, \sum_{x\in H} \|A(x)\|^2 < +\infty\right\}.$$

Set $x \otimes f$ for $x \in H$ and $f \in \mathcal{H}$ as

$$(x \otimes f)(y) = \begin{cases} f & x = y \\ 0 & x \neq y\,. \end{cases}$$

(Note that $l^2(H,\mathcal{H})$ is isomorphic to $l^2(H) \otimes \mathcal{H}$.) The application

$$y \mapsto \pi(y) \qquad \pi(y)(x \otimes f) = (x-y) \otimes W(y)f$$

is a representation of the CCR on the Hilbert space $l^2(H,\mathcal{H})$. π is equivalent to R. If a unitary $U : l^2(H,\mathcal{H}) \to l^2(H,\mathcal{H})$ is defined as

$$U(x \otimes f) = x \otimes W(x)f$$

then

$$U\,\pi(y) = (R(y) \otimes id)U \qquad (y \in H).$$

To prove our claim it is sufficient to find an isomorphism between $\mathcal{B}$ and the C*–algebra generated by $\{\pi(y) \,:\, y \in H\}$. We show that for any finite linear combination

$$\left\|\sum \lambda_i\, W(y_i)\right\| = \left\|\sum \lambda_i\, \pi(y_i)\right\| \tag{2.2}$$

holds.

Let $\hat{H}$ stand for the dual group of the discrete group H. $\hat{H}$ consists of characters of H and endowed by the topology of pointwise convergence forms a compact topological group. We consider the normalized Haar measure on $\hat{H}$. The spaces $l^2(H)$ and $L^2(\hat{H})$ are isomorphic by the Fourier transformation, which establishes the unitary equivalence between the above π and $\hat{\pi}$ defined below.

$$\hat{\pi}(y)\hat{A}(\chi) = \chi(y)W(y)\hat{A}(\chi) \qquad (y \in H\,,\;\; \chi \in \hat{H}\,,\;\; \hat{A} \in L^2(\hat{H},\mathcal{H}))$$

Hence

$$\left\|\sum \lambda_i\, \pi(y_i)\right\| = \left\|\sum \lambda_i\, \hat{\pi}(y_i)\right\|. \tag{2.3}$$

A closer look at the definition of $\hat{\pi}$ gives that $\hat{\pi}(y)$ is essentially a multiplication operator (by $\chi(y)W(y)$) and its norm is the sup norm. That is,

$$\left\|\sum \lambda_i\, \hat{\pi}(y_i)\right\| = \sup\left\{\left\|\sum \lambda_i\, \chi(y_i)W(y_i)\right\| \,:\, \chi \in \hat{H}\right\}. \tag{2.4}$$

Since the right hand side is the sup of a continuous function over $\hat{H}$, this sup may be taken over any dense set.

Let us set

$$G = \{\exp(2\mathrm{i}\,\sigma(x,\cdot) \,:\, x \in H\}.$$

Clearly, $G \subset \hat{H}$ is a subgroup. The following result is at our disposal (see (23.26) of [19]).

If $K \subset \hat{H}$ is a proper closed subgroup then there exists $0 \neq h \in H$ such that $k(h) = 1$ for every $k \in K$.

Assume that $\exp(2\mathrm{i}\,\sigma(x,y)) = 1$ for every $x \in H$. Then for every $t \in \mathbb{R}$ there exists an integer $l \in \mathbb{Z}$ such that $t\,\sigma(x,y) = l\,\pi$. This is possible if $\sigma(x,y) = 0$ (for every $x \in H$) and y must be 0. According to the above cited result of harmonic analysis the closure of G must be the whole $\hat{H}$.

Now we are in a position to complete the proof. For

$$\chi(\cdot) = \exp(2\mathrm{i}\,\sigma(x,\cdot)) \in G$$

we have

$$\begin{aligned}\left\|\sum \lambda_i\, \chi(y_i)W(y_i)\right\| &= \left\|W(x) \sum \lambda_i\, W(y_i)W(-x)\right\| = \\ &= \left\|\sum \lambda_i\, W(y_i)\right\|\end{aligned}$$

and this is the supremum in (4.4). Through (4.4) we arrive at (4.2). □

The previous theorem is due to Slawny [**Sl**]. We learnt from the proof that $\mathrm{CCR}(H,\sigma)$ has a representation on $l^2(H)$ given by (4.1). The subalgebra

$$\left\{\sum_{x\in H} \lambda(x)R(x) \,:\, \lambda \,:\, H \to \mathbb{C} \text{ has finite support}\right\}$$

is dense in $\mathrm{CCR}(H,\sigma)$ and there exists a state τ on $\mathrm{CCR}(H,\sigma)$ such that

$$\tau\left(\sum \lambda(x)R(x)\right) = \lambda(0)\,. \tag{2.5}$$

It is simple to verify that $\tau(ab) = \tau(ba)$. Therefore, τ is called the tracial state of $\mathrm{CCR}(H,\sigma)$. We can use τ to prove the following.

Proposition 2.3 *If $f, g \in H$ are different then*

$$\|W(f) - W(g)\| \geq \sqrt{2}.$$

Proof. For $h_1 \neq h_2$, we have $\tau(W(h_1)W(-h_2)) = 0$. Hence $\|W(f) - W(g)\|^2 \geq \tau((W(f) - W(g)^*(W(f) - W(g))) = 2$. □

It follows from the Proposition that the unitary group $t \mapsto W(tf)$ is never normcontinuous and the C*-algebra $\mathrm{CCR}(H, \sigma)$ can not be separable.

Slawny's theorem has also a few important consequences. Clearly for $(H_1, \sigma_1) \subset (H_2, \sigma_2)$ the inclusion $\mathrm{CCR}(H_1, \sigma_1) \subset \mathrm{CCR}(H_2, \sigma_2)$ must hold. (If H_1 is a proper subspace of H_2 then $\mathrm{CCR}(H_1, \sigma_2)$ is a proper subalgebra of $\mathrm{CCR}(H_2, \sigma_2)$.) If $T : H \to H$ is an invertible linear mapping such that

$$\sigma(f, g) = \sigma(Tf, Tg) \tag{2.6}$$

then it may be lifted into a *-automorphism of $\mathrm{CCR}(H, \sigma)$. Namely, there exists an automorphism γ_T of $\mathrm{CCR}(H, \sigma)$ such that

$$\gamma_T(W(f)) = W(Tf) \tag{2.7}$$

A simple example is the parity automorphism

$$\pi(W(f)) = W(-f) \qquad (f \in H)\,. \tag{2.8}$$

Let (H, σ) be a symplectic space. A real linear mapping $J : H \to H$ is called a complex structure if

(i) $J^2 = -I$,

(ii) $\sigma(Jf, f) \leq 0 \qquad (f \in H)$,

(iii) $\sigma(f, g) = \sigma(Jf, Jg) \qquad (f, g \in H)\,.$

If a complex structure J is given then H may be considered as a complex vectorspace setting

$$(t + is)f = tf + sJf \qquad (s, t \in \mathbb{R}\,,\;\; f \in H) \tag{2.9}$$

The definition

$$\langle f, g \rangle = \sigma(f, Jg) + i\,\sigma(f, g) \tag{2.10}$$

supplies us (a complex) inner product. So to have a symplectic space (over the reals) with a complex structure is equivalent to being given a complex inner product space.

Let J be a complex structure over (H, σ). The gauge automorphism

$$\gamma_\alpha(W(f)) = W(\cos\alpha\, f + J\, \sin\alpha\, f) \quad (\alpha \in [0, 2\pi]\,,\;\; f \in H) \tag{2.11}$$

is another example for lifting of a mapping into an automorphism.

We shall restrict ourselves mainly to C*-algebras associated to a nondegenerate symplectic space but degeneracy of the symplectic form appears in certain cases. Now this possibility will be discussed following the paper [25].

Let σ be (a possible degenerate) symplectic form on H. We write $\Delta(H, \sigma)$ for the free vectorspace generated by the symbols $\{W(h) : h \in H\}$. So $\Delta(H, \sigma)$ consists of formal finite linear combinations like

$$\sum \lambda_i\, W(h_i)\,.$$

We may endow $\Delta(H,\sigma)$ by a *-algebra structure by setting

$$W(h)^* = W(-h) \qquad (h \in H) \tag{2.12}$$

and

$$W(h)W(g) = \exp(\mathrm{i}\,\sigma(x,y))W(h+y) \qquad (h,y \in H) \tag{2.13}$$

On the *-algebra $\Delta(H,\sigma)$ we shall consider the so-called minimal regular norm (cf. [29], Ch. IV §18.3). We take all *-representations π of $\Delta(H,\sigma)$ by bounded Hilbert space operators and define

$$\|a\| = \sup\{\|\pi(a)\| \,:\, \pi \text{ is a representation}\} \qquad (a \in \Delta(H,\sigma)) \tag{2.14}$$

Another possibility is to take all positive normalized functionals (that is, states) φ on $\Delta(H,\sigma)$ and to introduce the norm

$$\|a\| = \sup\{\varphi(a^*a)^{1/2} \,:\, \varphi \text{ is a state}\} \qquad (a \in \Delta(H,\sigma)) \tag{2.15}$$

One can see that (4.14) and (4.15) determine the same norm, called the minimal regular norm. The completion of $\Delta(H,\sigma)$ with respect to $\|\cdot\|$ will be a C*–algebra and it is CCR(H,σ) by definition. It follows from Slawny's theorem that for nondegenerate σ the previous and the latter definitions coincide.

Now we study the extreme case when $\sigma \equiv 0$. Then $\Delta(H,\sigma)$ is commutative and a state φ of it corresponds to a positive-definite function F on the discrete abelian group H. We have

$$\varphi\left(\sum \overline{\lambda_i}\, W(h_i)^* \sum \lambda_j\, W(h_j)\right) \geq 0$$

for every $\lambda_i \in \mathbb{C}$ and $h_i \in H$ if and only if the function

$$F \,:\, h \mapsto \varphi(W(h)) \qquad (h \in H)$$

is positive-definite. Due to Bochner's theorem ([19], 33.1) there is a probability measure μ on the compact dual group $\hat{H}$ such that

$$F(h) = \int \chi(h)\, d\mu(\chi) \qquad (h \in H)\,.$$

Hence

$$\sup\left\{\varphi(a^*a)^{1/2} \,:\, \varphi \text{ is a state}\right\} = \sup\left\{\chi(a^*a)^{1/2} \,:\, \chi \in \hat{H}\right\},$$

where for $a = \sum \lambda_i\, W(h_i) \in \Delta(H,\sigma)$ $\quad \chi(a)$ (or $a(\chi)$) is defined as

$$\sum \lambda_i\, \chi(h_i)\,.$$

In this way every element a of $\Delta(H,\sigma)$ may be viewed to be a continuous function on $\hat{H}$ and

$$\|a\| = \sup\left\{|a(\chi)| \,:\, \chi \in \hat{H}\right\} \qquad (a \in \Delta(H,\sigma))\,.$$

$\Delta(H,\sigma)$ evidently separates the points of $\hat{H}$ and the Stone-Weierstrass theorem tells us that CCR(H,σ) is isomorphic to the C*-algebra of all continuous functions on the compact space $\hat{H}$.

The case of a vanishing symplectic form does not occur frequently, however, it may happen that $H = H_0 \oplus H_1$ and

$$\sigma\,(h_0 \oplus h_1\ ,\ h_0' \oplus h_1') = \sigma_1\,(h_1 \oplus h_1')$$

with a nondegenerate symplectic form σ_1 on H_1. Then the *-algebra $\Delta(H,\sigma)$ is the algebraic tensor product of $\Delta(H_0,0)$ and $\Delta(H_1,\sigma_1)$ and $\mathrm{CCR}(H,\sigma)$ will be

$$\mathrm{CCR}(H_0,0) \otimes \mathrm{CCR}(H_1,\sigma_1)\,. \tag{2.16}$$

(Note that since $\mathrm{CCR}(H_0,0)$ is commutative, the C*-norm on the tensor product is unique.)

Now we review briefly the general case. For a degenerate symplectic form σ we set

$$H_0 = \{x \in H\ :\quad \sigma(x,y) = 0 \quad \text{for every} \quad y \in H\}$$

for the kernel of σ. $\Delta(H_0,0)$ is the center of the *-algebra $\Delta(H,\sigma)$ and there exists a natural projection E given by

$$E\left(\sum_{x\in H} \lambda(x)W(x)\right) = \sum_{x\in H_0} \lambda(x)W(x) \tag{2.17}$$

and mapping $\Delta(H,\sigma)$ onto $\Delta(H_0,0)$. Having introduced the minimal regular norm we observe that $\mathrm{CCR}(H_0,0)$ is the center of $\mathrm{CCR}(H,\sigma)$ and E is a conditional expectation. The maximal *-ideals of $\mathrm{CCR}(H,\sigma)$ are in one-to-one correspondence with those of $\mathrm{CCR}(H_0,0)$. In particular, $\mathrm{CCR}(H,\sigma)$ is simple if and only if $H_0 = \{0\}$, that is, σ is nondegenerate. Concerning the details we refer to [25].

For a nondegenerate symplectic form Slawny's theorem provides readily that $\mathrm{CCR}(H,\sigma)$ is simple.

Chapter 3

Fock representation

3.1 The Fock state

Let $\mathcal{H}$ be a Hilbert space and CCR($\mathcal{H}$) be the corresponding CCR-algebra.

Theorem 3.1 *There is a state φ on the C^*-algebra CCR($\mathcal{H}$) such that*

$$\varphi(W(f)) = \exp(-\|f\|^2/2). \tag{3.1}$$

Proof. $\varphi(I) = 1$ follows from $f = 0$. The state φ exists if $\varphi(A^*A) \geq 0$ when A is a linear combination of Weyl operators. Assume that $A = \sum_i \lambda_i W(f_i)$. Then

$$\begin{aligned} A^*A &= \sum_i \overline{\lambda_i} W(-f_i) \sum_j \lambda_j W(f_j) \\ &= \sum_{i,j} \overline{\lambda_i}\lambda_j W(f_j - f_i) \exp \tfrac{1}{2}(-\langle f_i, f_j\rangle + \langle f_j, f_i\rangle) \end{aligned}$$

and

$$\begin{aligned} \varphi(A^*A) &= \sum_{i,j} \overline{\lambda_i}\lambda_j \exp \tfrac{1}{2}(-\langle f_j - f_i, f_j - f_i\rangle - \langle f_i, f_j\rangle + \langle f_j, f_i\rangle) \\ &= \sum_{i,j} \overline{\lambda_i}\lambda_j \exp \tfrac{1}{2}(-\|f_j\|^2 - \|f_i\|^2) \exp 2\langle f_j, f_i\rangle). \end{aligned}$$

This is positive (for all λ_i) if the matrx

$$(i, j) \mapsto \exp \tfrac{1}{2}(-\|f_j\|^2 - \|f_i\|^2) \exp 2\langle f_j, f_i\rangle)$$

is positive. This is the entry-wise product of the matrices

$$(i, j) \mapsto \exp \tfrac{1}{2}(-\|f_j\|^2 - \|f_i\|^2) \text{ and } (i, j) \mapsto \exp 2\langle f_j, f_i\rangle).$$

Due to the Hadamard theorem, it is enough to see that both are positive. The first one has the form X^*X, so it is positive. The second one is the entry-wise exponential of the positive Gram matrix $(\langle f_j, f_i\rangle)_{ij}$. This is positive as well.

The linear functional φ is defined on the linear combinations of the Weyl operators and it is positive. By continuity, it can be extended to the whole CCR($\mathcal{H}$). □

The state defined by (5.1) is called **Fock state**.

Next we perform the GNS-representation. $\mathcal{H}_\varphi$ is the Hilbert space generated by $\mathrm{CCR}(\mathcal{H})$ with the inner product $\langle A, B\rangle := \varphi(A^*B)$. The vector I is usually denoted by Φ and called **vacuum vector**. The representation $\pi_\varphi : \mathrm{CCR}(\mathcal{H}) \to B(\mathcal{H}_\varphi)$ is defined as

$$\pi_\varphi(B)A = BA \qquad (A, B \in \mathrm{CCR}(\mathcal{H})).$$

The represntation π_φ is the **Fock representation**.

The example in Chapter 1 corresponds to the Fock representation of $\mathrm{CCR}(\mathcal{H})$ when $\mathcal{H}$ has dimension 1. Lemma 1.4 gives the example

$$W(x + \mathrm{i}y) = \exp \mathrm{i}\sqrt{2}(xP + yQ).$$

The formula (1.20) shows that φ_0 is the vacuum vector and the Hilbert space $\mathcal{H}_\varphi$ is $L^2(\mathbb{R})$.

In the rest of this chapter $W(f)$ and $\pi_\varphi(W(f))$ will be identified.

3.2 Field operators

An important property of this representation that the one-parameter group $U_t := W(tf)$ of unitaries is weakly continuous, since the function

$$t \mapsto \langle W(g), W(tf)W(h)\rangle$$

is continuous for every $g, h \in \mathcal{H}$. Due to the Stone theorem, there is a self-adjoint operator $B(f)$ on $\mathcal{H}_\varphi$ such that

$$W(tf) = \exp(\mathrm{i}tB(f)) \qquad (t \in \mathbb{R}). \tag{3.2}$$

It follows from Proposition 4.3 that the **field operator** $B(f)$ must be unbounded. The vectors $W(g)\Phi$ are in the domain of $B(f)$ and more generally, in the domain of $B(f_1)B(f_2)\dots B(f_k)$. The expression $\varphi(B(f_1)B(f_2)\dots B(f_k))$ is defined as

$$\langle \Phi, B(f_1)B(f_2)\dots B(f_k)\Phi\rangle.$$

Proposition 3.2 *Then for $f, g \in \mathcal{H}$ and $t \in \mathbb{R}$ the following relations hold in the Fock representation.*

(i) $B(tf) = tB(f), \quad B(f + g) = B(f) + B(g).$

(ii) $[B(f), W(g)] = 2\sigma(f, g)W(g), \quad [B(f), B(g)] = -2i\sigma(f, g).$

(iii) $\varphi(B(f)B(g)) = \langle B(f)\Phi, B(g)\Phi\rangle = \langle f, g\rangle.$

Set

$$B^\pm(f) = \frac{1}{2}(B(f) \mp \mathrm{i}B(\mathrm{i}f))\,. \tag{3.3}$$

Then

$$B(f) = B^+(f) + B^-(f)) \tag{3.4}$$

and

$$[B^-(f), B^+(g)] = \langle g, f\rangle \qquad (f, g \in \mathcal{H})\,. \tag{3.5}$$

$B^+(f)$ is called **creation operator** and $B^-(f)$ is **annihilation operator**.

Lemma 3.3 $B^-(f)\Phi = 0$.

Lemma 3.4 *For $k \in \mathbb{Z}$ we have*

$$B^-(f)B^+(f)^k\Phi = k\|f\|^2 B^+(f)^{k-1}\Phi \qquad (f \in H)\,.$$

Proof. We apply induction. The case $k = 0$ is contained in the previous lemma. Due to the commutation relation (5.5) we have

$$\begin{aligned} B^-(f)B^+(f)^{k+1}\Phi &= (B^+(f)B^-(f) + \langle f, f\rangle)B^+(f)^k\Phi \\ &= (k-1)\|f\|^2B^+(f)^k\Phi + \|f\|^2B^+(f)^k\Phi\,. \end{aligned}$$

□

One obtains by induction again the following.

Proposition 3.5 *If $n, k \in \mathbb{N}$ and $f \in \mathcal{H}$, then*

$$B^-(f)^nB^+(f)^k\Phi = \begin{cases} 0 & \text{if} \quad n > k \\ \frac{k!}{(k-n)!}\|f\|^{2n}B^+(f)^{k-n}\Phi & \text{if} \quad n \le k\,. \end{cases}$$

Example 3.6 We assume that $\mathcal{H}$ is of one dimension. Fix a unit (basis) vector η in $\mathcal{H}$ and set

$$f_n = \frac{1}{\sqrt{n!}}B^+(\eta)^n\Phi \qquad (n \in \mathbb{Z}_+)\,. \tag{3.6}$$

Then $\{f_0, f_1, \ldots\}$ is an orthonormal basis in the Fock space. If we write a^+ for $B^+(\eta)$ and a for $B^-(\eta)$ then

$$a^+f_n = \sqrt{n+1}f_{n+1} \qquad af_n = \begin{cases} \sqrt{n}f_{n-1} & n \ge 1 \\ 0 & n = 0 \end{cases}$$

and

$$[a, a^+] = 1\,.$$

With the choice

$$q = \frac{1}{\sqrt{2}}(a + a^+) \quad \text{and} \quad p = \frac{i}{\sqrt{2}}(a^+ - a)$$

the Heisenberg commutation relation is satisfied.

The vector f_n is called n-particle vector in the physics literature. Transforming f_n into f_{n+1} the operator a^+ increases the number of particles. This is the origin of the term **creation operator**. The operator a **annihilates** in the similar sense.

Our present formulas are very similar to those in Chapter 1. The Fock space $\mathcal{K}$ can be identified with $L^2(\mathbb{R})$ if the vector f_n corresponds to the Hermite function $\varphi_n \in L^2(\mathbb{R})$. It is clear that the operators a and a^+ are the same.

We compute the coordinates of the vectors $W(z)\Phi$ in the basis $\{f_n : n \in \mathbb{Z}^+\}$. For the sake of simplicity we choose $\eta = 1$.

$$\langle W(z)\Phi, f_n\rangle \;=\; \langle \exp(iB^+(z) + iB^-(z))\Phi, f_n\rangle$$

$$
\begin{aligned}
&= \exp\left(-\frac{1}{2}[iB^+(z), iB^-(z)]\right) \langle \exp(iB^+(z))\Phi, f_n\rangle \\
&= \exp\left(-\frac{1}{2}|z|^2\right) \sum \frac{(iz)^m}{m!} \langle \sqrt{m!} f_m, f_n\rangle \\
&= \exp\left(-\frac{1}{2}|z|^2\right) \frac{(iz)^n}{\sqrt{n!}}
\end{aligned}
$$

Hence for any $z \in \mathbb{C}$ the associated exponential vector $e(z)$ is the sequence

$$
\left(1, iz, \frac{(iz)^2}{\sqrt{z!}}, \cdots, \frac{(iz)^n}{\sqrt{n!}}, \cdots\right) \equiv \sum_n \frac{(iz)^n}{\sqrt{n!}} f_n. \tag{3.7}
$$

□

3.3 Fock space

Let $\{\eta_i : i \in I\}$ be an orthonormal basis in the complex Hilbert space $\mathcal{H}$. We set

$$
\left|\eta_{i_1}^{n_1}; \eta_{i_2}^{n_2}; \ldots; \eta_{i_k}^{n_k}\right\rangle = \frac{1}{\sqrt{n_1! \ldots n_k!}} B^+(f_{i_1})^{n_1} \ldots B^+(f_{i_k})^{n_k} \Phi \,. \tag{3.8}
$$

So for every choice of different indices $i_1, i_2, \ldots, i_k$ in I and $n_1, n_2, \ldots, n_k \in \mathbb{N}$ we get to a unit vector in $\mathcal{K}$. The vectors

$$
\left|\eta_{i_1}^{n_1}; \ldots, \eta_{i_k}^{n_k}\right\rangle \quad \text{and} \quad \left|\eta_{j_1}^{m_1}; \ldots, \eta_{j_l}^{m_l}\right\rangle
$$

are different if $((n_1, i_1), \ldots, (n_k, i_k))$ is not a permutation of $((m_1, j_1), \ldots, (m_l, j_l))$ and in this case they are orthogonal. All such vectors form a canonical orthonormal basis in $\mathcal{K}$.

From $\varphi(B(f)B(g)) = \langle f, g\rangle$ we deduce

$$
\varphi(B^{\pm}(f)B^{\pm}(g)) = 0
$$

if $f \perp g$. If the sequence $f_1, f_2, \ldots, f_n$ in $\mathcal{H}$ has the property that any two vectors are orthogonal or identical then in the expansion (5.22) of

$$
\varphi(B^{\pm}(f_n)B^{\pm}(f_{n-1}) \ldots B^{\pm}(f_1))
$$

we may have a nonzero term if always identical vectors are paired together. We benefit from this observation in the next proposition.

Proposition 3.7 *Assume that $g_1, g_2, \ldots, g_k$ are pairwise orthogonal vectors in H. Then*

$$
B^+(g_1)^{m_1} B^+(g_2)^{m_2} \ldots B^+(g_k)^{m_k} \Phi
$$

and

$$
B^+(g_1)^{n_1} B^+(g_2)^{n_2} \ldots B^+(g_k)^{n_k} \Phi
$$

are orthogonal whenever $m_j \neq n_j$ for at least one $1 \leq j \leq k$.

Proof. Suppose that $m_1 \neq n_1$ and $n_1 > m_1$. The inner product of the above vectors is given by

$$\varphi\left(B^-(g_k)^{n_k}\dots B^-(g_1)^{n_1}B^+(g_1)^{m_1}\dots B^+(g_k)^{m_k}\right)$$

and equals to

$$\varphi\left(B^-(g_1)^{n_1}B^+(g_2)^{m_1}\right)\varphi\left(B^-(g_k)^{n_k}\dots B^-(g_2)^{n_2}B^+(g_2)^{m_2}\dots B^+(g_k)^{m_k}\right).$$

Here the first factor vanishes due to $n_1 > m_1$.

We are able to conclude also the formula

$$\|B^+(g_1)^{n_1}B^+(g_2)^{n_2}\dots B^+(g_k)^{n_k}\Phi\|^2 = n_1!n_2!\dots n_k! \tag{3.9}$$

provided that $\|g_1\| = \|g_2\| = \ldots = \|g_k\| = 1$.

Lemma 3.8 *For $g_1, g_2, \ldots, g_n, f \in H$ with $\|f\| = 1$ we have*

$$\|B(f)B(g_1)B(g_2)\dots B(g_n)\Phi\| \le 2\sqrt{n+1}\|B(g_1)\dots B(g_n)\Phi\|.$$

We consider the linear subspace spanned by the vectors $f, g_1, g_2, \ldots, g_n$ and take an orthonormal basis $f_1 = f, f_2, \ldots, f_k$. We may express $B(g_i)$ by creation and annihilation operators corresponding to the basis vectors and get

$$\eta = B(g_1)\dots B(g_n)\Phi = \sum \lambda(n_1,\ldots,n_k)B^+(f_1)^{n_1}\dots B^+(f_k)^{n_k}\Phi$$

(Here the summation is over the k-triples $(n_1, \ldots, n_k)$ such that $n_i \in \mathbb{Z}_+$ and $\sum n_i \le n$.) Since

$$\|B(f)\eta\| \le \|B^+(f_1)\eta\| + \|B^-(f_1)\eta\|$$

it suffices to show that

$$\|B^\pm(f_1)\eta\|^2 \le (n+1)\|\eta\|^2.$$

Now we estimate as follows.

$$\begin{aligned}
\|B^+(f_1)\eta\|^2 &= \|\sum \lambda(n_1,\ldots,n_k)B^+(f_1)^{n_1+1}B^+(f_2)^{n_2}\dots B^+(f_k)^{n_k}\Phi\|^2 \\
&= \sum \|\lambda(n_1,\ldots,n_k)B^+(f_1)^{n_1+1}B^+(f_2)^{n_2}\dots B^+(f_k)^{n_k}\Phi\|^2 \\
&= \sum (n_1+1)\|\lambda(n_1,\ldots,n_k)B^+(f_1)^{n_1}\dots B^+(f_k)^{n_k}\Phi\|^2 \\
&\le (n+1)\|B(g_1)\dots B(g_n)\Phi\|^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|B^-(f_1)\eta\|^2 &= \|\sum_{n_1\ge 1} \lambda(n_1,\ldots,n_k)n_1B^+(f_1)^{n_1-1}B^+(f_2)^{n_2}\dots B^+(f_k)^{n_k}\Phi\|^2 \\
&= \sum n_1^2|\lambda(n_1,\ldots,n_k)|^2\|B^+(f_1)^{n_1-1}B^+(f_2)^{n_2}\dots B^+(f_k)^{n_k}\Phi\|^2 \\
&= \sum n_1\|\lambda(n_1,\ldots,n_k)B^+(f_1)^{n_1}B^+(f_2)^{n_2}\dots B^+(f_k)^{n_k}\Phi\|^2 \\
&\le n\sum \|\lambda(n_1,\ldots,n_k)B^+(f_1)^{n_1}B^+(f_2)^{n_2}\dots B^+(f_k)^{n_k}\Phi\|^2 = n\|\eta\|^2.
\end{aligned}$$

Lemma 5.4 and the explicit norm expression (5.9) have been used.

Let the Fock representation act on a Hilbert space $\mathcal{H}$ containing the vacuum vector Φ. The linear span of the vectors $B(g_1)B(g_2)\dots B(g_n)\Phi$ $(g_1, g_2, \dots, g_n \in H,\ n \in \mathbb{N})$ and $B^+(g_1)B^+(g_2)\dots B^+(g_n)\Phi$ $(g_1, g_2, \dots,\ g_n \in H,\ n \in \mathbb{N})$ coincide. It will be denoted by $\mathcal{D}_B$. So far it is not clear whether $\mathcal{D}_B$ is complete. This is what we are going to show.

Let A be a linear operator on a Hilbert space $\mathcal{K}$. A vector $\xi \in \mathcal{K}$ is called **entire analytic** (for A) if ξ is in the domain of A^n for every $n \in \mathbb{N}$ and

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \|A^k \xi\| < +\infty$$

for every $t > 0$. If ξ is an entire analytic vector then $\exp(zA)\xi$ makes sense for every $z \in \mathbb{C}$ and it is an entire analytic function of z.

Theorem 3.9 *$\mathcal{D}_B$ consists of entire analytic vectors for $B(f)$ $(f \in H)$.*

Proof. Let $\xi = B(f_1)B(f_2)\dots B(f_n)\Phi \in \mathcal{D}_B$. By a repeated application of Lemma 5.8 we have

$$\|B(f)^k \xi\| \le 2^k \sqrt{\frac{(n+k)!}{n!}} \|\xi\|$$

and it is straightforward to check that the power series

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \|B(f)^k \xi\|$$

converges for every t. Since the entire analytic vectors form a linear subspace, the proof is complete. □

Due to Theorem 5.9 every vector $W(f)\Phi = \exp(iB(f))\Phi$ can be approximated (through the power series expansion of the exponential function) by elements of $\mathcal{D}_B$. This yields, immediately that $\mathcal{D}_B$ is dense in $\mathcal{H}$. According to Nelson's theorem on analytic vectors (see [34] X.39), $\mathcal{D}_B$ is a core for $B(f)$ $(f \in H)$, in other words, $B(f)$ is the closure of its restriction to $\mathcal{D}_B$. It follows also that $\mathcal{D}_B$ is core for $B^{\pm}(f)$ and $B^-(f)^* = B^+(f)$.

Theorem 3.10 *The Fock representation is irreducible.*

Proof. We have to show that for any $0 \ne \eta \in \mathcal{H}$ the closed linear subspace $\mathcal{H}_1$ generated by $\{W(f)\eta : f \in H\}$ is $\mathcal{H}$ itself. Let $\mathcal{M}$ be the von Neumann algebra generated by the unitaries $\{W(f) : f \in H\}$ in $B(\mathcal{H})$. Clearly, $\mathcal{M}\eta \subset \mathcal{H}_1$.

We consider a canonical basis in $\mathcal{H}$ consisting of vectors (5.8). $\eta \in \mathcal{H}$ has an expansion as (countable) linear combinations of basis vectors. Assume that a vector

$$|f_1^{n_1}; f_2^{n_2}; \dots; f_k^{n_k}\rangle \tag{3.10}$$

has a nonzero coefficient.

The operator

$$B^+(f_1)B^-(f_1)\dots B^+(f_k)B^-(f_k) \tag{3.11}$$

is selfadjoint and (4.5) is its eigenvector with eigenvalue $n_1 + n_2 + \ldots + n_k$. Since (5.11) is affiliated with $\mathcal{M}$, its spectral projections are in $\mathcal{M}$. In this way we conclude that the vector (5.10) lies in $\mathcal{H}_1$.

It is easy to see that

$$B^{\pm}(f)\mathcal{H}_1 \subset \mathcal{H}_1$$

for every $f \in H$. By application of the annihilation operators $B^-(f_i)$ $(1 \le i \le k)$ we obtain that the cyclic (vacuum) vector Φ is in $\mathcal{H}_1$. Therefore, $\mathcal{H}_1 = \mathcal{H}$ must hold. □

Next we introduce some vectors of special importance by means of the Weyl operators. For $f \in H$ set

$$e(f) = \exp\left(\tfrac{1}{2}\|f\|^2\right) W(f)\Phi\,, \tag{3.12}$$

which is called **exponential vector**. One may compute that

$$\langle e(f), e(g)\rangle = \exp\langle g, f\rangle \qquad (f, g \in H) \tag{3.13}$$

Proposition 3.11 *$\{e(f) : f \in H\}$ is a linearly independent complete subset of $\mathcal{H}$.*

Proof. We use the fact that the family $\{e^{tx} : x \in \mathbb{R}\}$ of exponential functions is linearly independent.

Let $f_1, f_2, \ldots, f_n \in H$ be a sequence of different vectors and assume that $\sum \lambda_i e(f_i) = 0$. We choose a vector $g \in H$ such that the numbers

$$\mu_i = \langle f_i, g\rangle \quad (1 \le i \le n)$$

are distinct. For any $t \in \mathbb{R}$ we have

$$0 = \langle e(tg), \sum \lambda_i e(f_i)\rangle = \sum \lambda_i \exp(t\langle f_i, g\rangle)$$

and we may conclude that $\lambda_i = 0$ for every $1 \le i \le n$.

Due to the cyclicity of the vacuum vector Φ the set $\{e(f) : f \in H\}$ is complete. A little bit more is true. The norm expression

$$\|e(f) - e(g)\|^2 = \exp(\|f\|^2) + \exp\|g\|^2 - 2\mathrm{Re}\ \exp\langle f, g\rangle \tag{3.14}$$

tells us that the mapping $f \mapsto e(f)$ is norm continuous. Therefore $\{e(f) : f \in S\}$ is complete whenever S is a dense subset of $\mathcal{H}$. □

Example 3.12 Assume that $\mathcal{H}$ has two dimension with orthogonal unit vectors η_1 and η_1. We use the notation $B^+(\eta_i) = B_i^+$ and $B^-(\eta_i) = B_i^-$ $(i = 1, 2)$. The vectors

$$f_i^{(1)} = B_i^+\Phi \qquad (i = 1, 2)$$

are orthonormal and orthogonal to Φ:

$$\langle \Phi, f_i^{(1)}\rangle = \langle B_i^-\Phi, \Phi\rangle = 0 \quad \text{due to Lemma 5.3,}$$

$$\langle f_i^{(1)}, f_j^{(1)}\rangle = \langle B_i^+\Phi, B_j^+\Phi\rangle = \langle B_i\Phi, B_j\Phi\rangle = \langle \eta_i, \eta_j\rangle \quad \text{due to (iii) in Proposition 5.2.}$$

In the previous formalism it was

$$f_i^{(1)} = |\eta_i\rangle \qquad (i = 1, 2).$$

The next subspace has 3 dimension:

$$|\eta_1^2\rangle = \frac{1}{\sqrt{2!}}(B_1^+)^2\Phi, \quad |\eta_2^2\rangle = \frac{1}{\sqrt{2!}}(B_2^+)^2\Phi, \quad |\eta_1, \eta_2\rangle = B_1^+ B_2^+ \Phi.$$

The subspaces of the Fock space are antisymmetric tensor powers of the Hilbert space $\mathcal{H}$ which is two dimensional now. Thext subspace is the third power of $\mathcal{H}$:

$$|\eta_1^3\rangle = \frac{1}{\sqrt{3!}}(B_1^+)^3\Phi, \quad |\eta_1^2, \eta_2\rangle = \frac{1}{\sqrt{2!}}(B_1^+)^2 B_2^+\Phi,$$

$$|\eta_1, \eta_2^2\rangle = \frac{1}{\sqrt{2!}}B_1^+(B_2^+)^2\Phi, \quad |\eta_2^3\rangle = \frac{1}{\sqrt{3!}}(B_2^+)^3\Phi.$$

3.4 The positivity condition

To determine a state on $\mathrm{CCR}(H, \sigma)$, it is enough to give the values on the unitaries $W(f)$ $(f \in \mathcal{H})$. When the Fock state was introduced, an argument was required to show the existence. This will be extended now.

Let X be an arbitrary (nonempty) set. A function $F : X \times X \to \mathbb{C}$ is called a **positive definite kernel** if and only if

$$\sum_{j,k=1}^{n} c_j\, \overline{c}_k\, F(x_j, x_k) \geq 0$$

for all $n \in \mathbb{N}$, $\{x_1, x_2, \dots, x_n\} \subset X$ and $\{c_1, c_2, \dots, c_n\} \subset \mathbb{C}$. The product of positive definite kernels is positive definite. This statement is equivalent to the **Hadamard theorem** which says that the entry-wise product of positive matrices is positive.

Proposition 3.13 *Let (H, σ) be a symplectic space and $G : H \to \mathbb{C}$ a function. There exists a state φ on CCR(H, σ) such that*

$$\varphi(W(f)) = G(f) \qquad (f \in H)$$

if and only if $G(0) = 1$ and the kernel

$$(f, g) \mapsto G(f - g)\, \exp(-\mathrm{i}\, \sigma(f, g))$$

is positive definite.

Proof. For $x = \sum c_j\, W(f_j)$ we have

$$xx^* = \sum_{j,k} c_j\, \overline{c}_k\, W(f_j - f_k) e^{-i\sigma(f_j, f_k)}\,.$$

Since $\varphi(xx^*) \geq 0$ should hold, we see that the positivity condition is necessary.

On the other hand, the positivity condition allows us to define a positive functional on the linear hull of the Weyl operators and a continuous extension to $\mathrm{CCR}(H, \sigma)$ supplies a state. $\square$

Lemma 3.14 *Let (H,σ) be a symplectic space. (It might be degenerate.) If $\alpha(\,\cdot\,,\,\cdot\,)$ is a positive symmetric bilinear form on H then the following conditions are equivalent.*

(i) The kernel $(f,g)\mapsto \alpha(f,g)-\mathrm{i}\,\sigma(f,g)$ is positive definite.

(ii) $\alpha(z,z)\,\alpha(x,x)\ge \sigma(z,x)^2$ for every $x,z\in H$.

Proof. Both condition (i) and (ii) hold on H if and only if they hold on all finite dimensional subspaces. Hence we may assume that H is of finite dimension.

If $\alpha(x,x)=0$ then both condition (i) and (ii) imply that $\sigma(x,y)=0$ for every $y\in H$. Due to possible factorization we may assume that α is strictly positive and it will be viewed as an inner product.

There is an operator Q such that

$$\sigma(x,y)=\alpha(Qx,y)\qquad (x,y\in H)\,.$$

and $Q^*=-Q$ follows from $\sigma(x,y)=-\sigma(y,x)$. According to linear algebra in a certain basis the matrix of Q has a diagonal form $\mathrm{Diag}\,(A_1,A_2,\dots,A_k)$, where A_i is a 1×1 0-matrix or

$$A_i=\begin{pmatrix} 0 & a_i\\ -a_i & 0\end{pmatrix}.$$

(The first possibility occurs only if σ is degenerate.) It is easy to see that condition (i) is equivalent to $|a_i|\le 1$ and so is condition (ii). □

Theorem 3.15 *Let (H,σ) be a symplectic space and $\alpha: H\times H\to\mathbb{R}$ a symmetric positive bilinear form such that*

$$\sigma(f,g)^2\le \alpha(f,f)\alpha(g,g)\qquad (f,g\in H)\,. \tag{3.15}$$

Then there exists a state φ on $CCR(H,\sigma)$ such that

$$\varphi(W(f))=\exp\left(-\tfrac{1}{2}\alpha(f,f)\right)\qquad (f\in H)\,. \tag{3.16}$$

Proof. We are going to apply Proposition 5.13. Due to the positivity condition

$$\begin{aligned}
&\sum_{j,k} c_j\overline{c_k}\exp\left(-\tfrac{1}{2}\alpha(f_j-f_k,f_j-f_k)-\mathrm{i}\sigma(f_j,f_k)\right)\\
&\quad=\sum_{j,k}\left(c_j\exp\left(-\tfrac{1}{2}\alpha(f_j,f_j)\right)\right)\left(\overline{c}_k\exp\left(-\tfrac{1}{2}\alpha(f_k,f_k)\right)\right)\\
&\qquad\times\exp(\alpha(f_j,f_k)-\mathrm{i}\sigma(f_j,f_k))\\
&\quad=\sum_{j,k} b_j\overline{b}_k\exp(\alpha(f_j,f_k)-i\sigma(f_j,f_k))
\end{aligned}$$

should be shown to be nonnegative. According to Lemma 5.14

$$(\alpha(f_j,f_k)-i\sigma(f_j,f_k))_{j;k}$$

is positive definite and entrywise exponentiation preserves this property. □

A state φ on $\mathrm{CCR}(H,\sigma)$ determined in the form (5.16) is called **quasifree**.

A state is **regular** if in the GNS-representation the field operators exist.

Proposition 3.16 *Let φ be a state on CCR(H,σ). If*

$$\lim_{t\to 0}\varphi(W(tf)) = 1 \qquad (f \in H),$$

then φ is regular.

Proof. We set $G(f) = \varphi(W(f))$ $(f \in H)$. According to Proposition 5.13 the matrix

$$\begin{bmatrix} 1 & G(-f_1) & G(-f_2) \\ G(f_1) & 1 & G(f_1 - f_2)\exp(-i\sigma(f_1, f_2)) \\ G(f_2) & G(f_2 - f_1)\exp(i\sigma(f_1, f_2)) & 1 \end{bmatrix}$$

is positive definite. From this we obtain

$$|G(f_2) - G(f_1)| \le 4|1 - G(f_2 - f_1)\exp(-i\sigma(f_2, f_1))|\,. \tag{3.17}$$

Combining (5.17) with the hypothesis we arrive at the continuity of the function

$$t \mapsto G(tf + g) \qquad (t \in \mathbb{R})$$

for every $f, g \in H$. Let $(\pi_\varphi, \mathcal{H}_\varphi, \Phi)$ stand for the GNS-triple. We verify by computation the continuity of the function

$$t \mapsto \langle \pi_\varphi(W(tf))\pi_\varphi(W(g_1))\Phi, \pi_\varphi(W(g_2))\Phi\rangle \qquad (t \in \mathbb{R})$$

and the regularity of φ is proven. □

It folows that a quasifree state is regular.

3.5 Analytic states

A state φ on CCR(H,σ) is said to be **analytic** if the numerical function

$$t \mapsto \varphi(W(tf)) \qquad (t \in \mathbb{R})$$

is analytic. Quasifree states are obviously analytic.

Assume that π is a regular representation of CCR(H,σ). The **field operator** $B(g)$ is obtained by differentiation of the function

$$t \mapsto \pi(W(tg))\eta = \exp(itB(g))\eta \qquad (t \in \mathbb{R})\,. \tag{3.18}$$

More precisely, if (5.18) is weakly differentiable at $t = 0$ and the derivative is $\xi \in \mathcal{H}_\varphi$, then η is in the domain of $B(g)$ and

$$\mathrm{i}B(g)\eta = \xi\,.$$

Proposition 3.17 *Let φ be an analytic state on CCR(H,σ) with GNS-triple $(\pi_\varphi, \mathcal{H}_\varphi, \Phi)$. Then $\pi_\varphi(W(g))\Phi$ is in the domain of*

$$B(f_n)B(f_{n-1})\dots B(f_1)$$

for every $g, f_1, f_2, \dots, f_n \in H$ and $n \in \mathbb{N}$.

Proof. We apply induction and suppose that

$$\eta = B(f_{n-1}) \dots B(f_1)\pi_\varphi(W(g))\Phi$$

makes sense. For the sake of simpler notation we omit π_φ in the proof.

It suffices to show that

$$\lim_{t\to 0} t^{-1}\langle (W(f_n) - I)\eta, \xi\rangle = F(\xi) \tag{3.19}$$

exists if ξ is in a dense subset $\mathcal{D}_0$ of $\mathcal{H}_\varphi$ and $|F(\xi)| \le C\|\xi\|$. This ensures that

$$t \mapsto W(tf_n)\eta$$

is differentiable in the weak sense. Since for $\xi = W(h)\Phi$ the limit in (5.19) equals to

$$(-\mathrm{i})^n \frac{\partial}{\partial t} \frac{\partial^{n-1}}{\partial t_{n-1}\partial t_{n-2}\dots\partial t_1} \varphi(W(-h)W(tf_n)\dots W(t_1 f_1)W(g))$$

at the point $t = t_{n-1} = t_{n-2} = \dots = t_1 = 0$, the function F is defined on the linear hull $\mathcal{D}_W$ of the vectors

$$\{W(h)\Phi : h \in H\}\,.$$

F is linear on $\mathcal{D}_W$ and by differentiation one can see that

$$C = \lim_{t\to 0} \frac{1}{t}\|(W(tf_n) - I)\eta\|$$

exists and it fulfils $|F(\xi)| \le C\|\xi\|$ for $\xi \in \mathcal{D}_W$. □

Although $B(f) \notin \mathrm{CCR}(H,\sigma)$, it will be rather convenient to write $\varphi(B(f_n)B(f_{n-1})\dots B(f_1))$ instead of $\langle B(f_n)B(f_{n-1})\dots B(f_1)\Phi, \Phi\rangle$. We shall keep also the notation $\mathcal{D}_W$ from the above proof. Remember that $\mathcal{D}_W$ as well as the superset Hilbert space $\mathcal{H}_\varphi$ depend on the state φ even if it is excluded from the notation.

Proposition 3.18 *Let φ be an analytic state on CCR(H,σ). Then for $f, g \in H$ and $t \in \mathbb{R}$ the following relations hold on $\mathcal{D}_W$.*

(i) $B(tf) = tB(f), \quad B(f+g) = B(f) + B(g).$

(ii) $[B(f), W(g)] = 2\sigma(f,g)W(g), \quad [B(f), B(g)] = -2i\sigma(f,g).$

Proof. The relations are deduced by derivation from (the Weyl form of) the CCR. □

3.6 Quasifree states

Recall that a qusifree state is defined by the formula

$$\varphi(W(f)) = \exp(-\tfrac{1}{2}\alpha(f,f)),$$

where the real bilinear form α satisfies the positivity condition (5.15). A qusifree state is analytic and derivation yields

$$\varphi(B(f)B(g)) = \alpha(f,g) - \mathrm{i}\sigma(f,g). \tag{3.20}$$

Proposition 3.19 *Let φ be a quasifree state on CCR(H,σ) given by (5.16) and $f_1, f_2, \dots, f_n \in H$. Then*

$$\varphi(B(f_n)B(f_{n-1})\dots B(f_1)) = 0$$

if n is odd. For an even n we have

$$\varphi(B(f_n)B(f_{n-1})\dots B(f_1)) = \sum \prod_{m=1}^{n/2} \left(\alpha(f_{k_m}, f_{j_m}) - i\sigma(f_{k_m}, f_{j_m})\right),$$

where the summation is over all partitions $\{H_1, H_2, \dots, H_{n/2}\}$ of $\{1, 2, \dots, n\}$ such that $H_m = \{j_m, k_m\}$ with $j_m < k_m$ $(m = 1, 2, \dots, n/2)$.

Proof. We benefit from the formula

$$\varphi(B(f_n)B(f_{n-1})\dots B(f_1)) = (-i)^n \frac{\partial^n}{\partial_n \dots \partial_1}\varphi(W(t_n f_n)\dots W(t_1 f_1)).$$

Since we have

$$\begin{aligned} &W(t_n f_n)W(t_{n-1}f_{n-1})\dots W(t_1 f_1) \\ &\quad = W(f_n f_n + t_{n-1}f_{n-1} + \dots + t_1 f_1) \times \exp i\left(\sum_{l>k} t_l t_k \sigma(f_l, f_k)\right) \end{aligned}$$

(5.16) yields

$$\begin{aligned} \varphi(W(t_n f_n)\dots W(f_1 f_1)) = \exp\left(-\frac{1}{2}\sum_{m=1}^{n} t_m^2 \alpha(f_m, f_m)\right) \\ \exp\left(\sum_{l>k} t_l t_k(-\alpha(f_l, f_k) + i\sigma(f_l, f_k))\right). \end{aligned} \tag{3.21}$$

What we need is the coefficient of $t_1 t_2 \dots t_n$ in the power series expansion. Such term comes only from the second factor of (5.21) and only in the case of an even n. In the claim it is described exactly the possibilities for getting $t_1 t_2 \dots t_n$ as a product of factors $t_l t_k$ $(l > k)$. □

By means of (5.20) we have also

$$\varphi(B(f_n)B(f_{n-1})\dots B(f_1)) = \sum \prod \varphi\left(B(f_{k_m})B(f_{j_m})\right), \tag{3.22}$$

were summation and product are similar to those in Proposition 5.19. The expression (5.22) makes clear that the value of a quasifree state φ on any polynomial of field operators is completely determined by the **two-point-functions** $\varphi(B(f)B(g))$ $(f, g \in H)$.

3.7 Purification

Let H be a real Hilbert space with inner product $(\cdot\,,\,\cdot)$ and let σ be a nondegenerate symplectic form on H such that

$$|\sigma(f,g)|^2 \le (f,f)(g,g) \qquad (f, g \in H) \tag{3.23}$$

holds. There exists a contraction D on H such that

$$\sigma(f,g) = (Df,g) \qquad (f,g \in H)\,. \tag{3.24}$$

Evidently $D^* = -D$. If $Df = 0$ then due to the nondegeneracy of σ $f = 0$ and hence D is invertible. Consider the polar decomposition

$$D = J|D|\,. \tag{3.25}$$

The property $D^* = -D$ gives that

$$J|D|J^* = -J^2|D|$$

and the uniqueness of the polar decomposition (applied for the positive operator $J|D|J^*$) guarantees that

$$-J^2 = I \quad \text{and} \quad J|D| = |D|J\,. \tag{3.26}$$

The state space of a C*-algebra is a compact convex subset of the dual space if it is endowed with the weak topology. A state is called **pure** if it is an extremal point of the state space.

Proposition 3.20 *Let φ be a quasifree state on $CCR(H,\sigma)$ so that*

$$\varphi(W(h)) = \exp\left(-\tfrac{1}{2}(h,h)\right) \qquad (h \in H),$$

If φ is pure then $|D|$ (given by (5.25) is the identity.

Proof. We shall argue by contradiction. Assume that there exists $f \in H$ such that

$$(|D|f,f) = 1 \quad \text{and} \quad (|D|^{-1} - I)^{1/2} f \neq 0\,. \tag{3.27}$$

Set $L = |D|^{1/2}(|D|^{-1} - I)^{1/2}$ and note that L is a contraction. We define a symmetric bilinear form as

$$S(g_1,g_2) = (g_1,g_2) - (Lg_1,|D|^{1/2}f)\cdot(Lg_2,|D|^{1/2}f)\frac{(Lf,|D|^{1/2}f)^2}{(Lf,Lf)}$$

and show that

$$S(g,g) \geq (|D|g,g) \qquad (g \in H) \tag{3.28}$$

or equivalently

$$((I - |D|)g,g)(Lf,Lf) \geq (Lg,|D|^{1/2}f)^2(Lf,|D|^{1/2}f)^2\,. \tag{3.29}$$

(5.29) is a consequence of the Schwarz inequality :

$$\begin{aligned}
(Lf,|D|^{1/2}f)^2 &\leq (Lf,Lf)(|D|f,f) = (Lf,Lf)\\
(Lg,|D|^{1/2}f)^2 &\leq (L^2g,g)(|D|f,f) = ((I - |D|)g,g)\,.
\end{aligned}$$

By means of (5.24) and (5.25) we infer from (5.28) that

$$|\sigma(h,g)|^2 = (J|D|h,g)^2 \leq (|D|h,h)(|D|g,g) \leq S(h,h)S(g,g)\,.$$

Now Theorem 5.15 tells us that there is a (quasifree) state ω on $CCR(H,\sigma)$ such that

$$\omega(W(h)) = \exp\left(-\frac{1}{2}S(h,h)\right). \qquad (h \in H)$$

We can see from Proposition 5.13 that if ω is any state of $CCR(H,\sigma)$ and F is a linear functional on H then there exists a state ω_F such that

$$\omega_F(W(h)) = \omega(W(h))\exp(iF(h)).$$

Writing a for $(Lf,|D|^{1/2}f)(Lf,Lf)^{-1/2}$ we set a state ω_λ for $\lambda \in \mathbb{R}$ as follows.

$$\omega_\lambda(W(h)) = \exp\left(-\frac{1}{2}S(h,h) + i\lambda(Lh,|D|^{1/2}f)a\right) \quad (h \in H).$$

With the shorthand notation b for $(Lh,|D|^{1/2}f)$ we have

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{e^{-\lambda^2/2}}{(2\pi)^{1/2}}\omega_\lambda(W(h))d\lambda &= e^{-\frac{1}{2}(h,h)}(2\pi)^{-1/2} \\ &\quad\times\int_{-\infty}^{\infty}\exp(-\lambda^2/2+b^2a^2/2+iba)d\lambda \\ &= e^{-\frac{1}{2}(h,h)}(2\pi)^{-1/2} \\ &\quad\times\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}(\lambda+iab)^2\right)d\lambda \\ &= e^{-\frac{1}{2}(h,h)}\end{aligned}$$

and this means that

$$\varphi = (2\pi)^{-1/2}\int_{-\infty}^{\infty}\exp(-\lambda^2/2)\omega_\lambda d\lambda.$$

This decomposition is in contradiction with the starting assumption on φ. Hence the proof has been completed. □

Theorem 3.21 *Let the quasifree state φ defined on $CCR(H,\sigma)$ be given by a complete inner product $\alpha(\,\cdot\,,\,\cdot\,)$ as*

$$\varphi(W(h)) = \exp\left(-\tfrac{1}{2}\alpha(h,h)\right).$$

Then φ is pure if and only if it is a Fock state.

This result makes Proposition 5.20 more complete. Remember that a state on a C*-algebra is pure if and only if the corresponding GNS-representation is irreducible (see [6], Thm. 2.3.19). Theorem 5.10 tells us that Fock states are pure and Proposition 5.20 yields that the other states are not so.

Now we are going to see that every quasifree state is a restriction of a Fock state of a bigger CCR-algebra.

Theorem 3.22 *Let $H_2 = H \oplus H$ be the direct sum Hilbert space and set a contraction D_2 of H_2 by the matrix*

$$D_2 = \begin{bmatrix} D & J\sqrt{I+D^2} \\ J\sqrt{I+D^2} & -D \end{bmatrix}. \tag{3.30}$$

Then the bilinear form

$$\sigma_2(W(f_2)) = \exp\left(-\frac{1}{2}\|f_2\|^2\right) \qquad (f_2, g_2 \in H_2)$$

is a symplectic form and the quasifree state

$$\varphi_2(W(f_2)) = \exp\left(-\frac{1}{2}\|f_2\|^2\right) \qquad (f_2 \in H_2) \tag{3.31}$$

on $CCR(H_2, \sigma_2)$ is a Fock state.

Proof. The proof is rather straightforward. We recall the relations

$$JD = DJ\ ,\ D^* = -D\ ,\ J = J^*\ ,\ J^2 = -id$$

These give that

$$D_2^* = -D_2 \quad \text{and} \quad D_2^2 = -id\,,$$

in other words, D_2 is a skewadjoint unitary. Hence σ_2 is an antisymmetric form and (4.9) defines a quasifree state. By the definition at the end of Chapter 3, φ_2 is a Fock state. □

Since

$$\sigma_2(f \oplus 0, f' \oplus 0) = (D_2(f \oplus 0), f' \oplus 0) = (Df, f') = \sigma(f, f')$$

the mapping

$$W(f) \mapsto W(f \oplus 0) \qquad (f \in H)$$

gives rise to an embedding of $CCR(H, \sigma)$ into $CCR(H_2, \sigma_2)$. Fock states are pure and that is the reason why the procedure described in Theorem 4.9 is called purification. Due to the direct sum $H_2 = H \oplus H$, doubling is another used term.

Purification is a standard way to reduce assertions on arbitrary quasifree states to those on Fock states. For example, we have

Corollary 3.23 *For an arbitrary quasifree state φ the linear manifold $\mathcal{D}_B^\varphi$ is dense in the GNS Hilbert space $\mathcal{H}_\varphi$ and consists of entire analytic vectors for every field operator $B_\varphi(f)$ ($f \in H$).*

3.8 Exercises

1. Check that in formula (1.10) $a^* = a^+$ holds.
2. Compute the entropy of the density (1.23).
3. Check that
$$a = \frac{Q + \mathrm{i}P}{\sqrt{2}}.$$

Chapter 4

The C*-algebra of the canonical commutation relation

If $\mathcal{H}$ is a complex Hilbert space then $\sigma(f,g) = \operatorname{Im}\langle f,g\rangle$ is a nondegenerate symplectic form on the real linear space $\mathcal{H}$. (Symplectic form means $\sigma(x,y) = -\sigma(y,x)$.) (H,σ) will be a typical notation for a Hilbert space and it will be called symplectic space.

Let (H,σ) be a symplectic space. The C*-algebra of the **canonical commutation relation** over (H,σ), written as CCR(H,σ), is by definition a C*-algebra generated by elements $\{W(f) : f \in H\}$ such that

(i) $W(-f) = W(f)^* \qquad (f \in H)$,

(ii) $W(f)W(g) = \exp(\mathrm{i}\,\sigma(f,g))\,W(f+g) \qquad (f,g \in H)$.

Condition (ii) tells us that $W(f)W(0) = W(0)W(f) = W(f)$. Hence $W(0)$ is the unit of the algebra and it follows that $W(f)$ is a unitary for every $f \in H$.

The typical example comes from Lemma 1.4, see (1.18) and (1.19).

Example 4.1 If $f,g \in \mathcal{H}$ are orthogonal, then

$$W(f)W(g) = W(f+g) = W(g+f) = W(g)W(f).$$

It follows that for $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, CCR$(\mathcal{H})$ is isomorphic to CCR$(\mathcal{H}_1) \otimes$ CCR$(\mathcal{H}_2)$. $\square$

Theorem 4.2 *For any symplectic space (H,σ) the C*-algebra CCR(H,σ) exists and it is unique up to isomorphism.*

Proof. To establish the existence will be easier than proof of the uniqueness. Consider H as a discrete abelian group (with the vectorspace addition).

$$l^2(H) = \left\{ F : H \to \mathbb{C} : \sum_{x\in H} |F(x)|^2 < +\infty \right\}$$

is a Hilbert space. (Any element of $l^2(H)$ is a function with countable support.) Setting

$$(R(x)F)(y) = \exp(i\,\sigma(y,x))F(x+y) \qquad (x,y \in H) \tag{4.1}$$

we get a unitary $R(x)$ on $l^2(H)$ and one may check that

$$R(x_1)R(x_2) = \exp(i\,\sigma(x_1,x_2)R(x_1+x_2).$$

The norm closure of the set

$$\left\{\sum_{i=1}^{n} \lambda_i\, R(x_i) \quad : \quad \lambda_i \in \mathbb{C}\,,\ 1 \le i \le n\,,\ n \in \mathbb{N}\,,\ x_i \in H\right\}$$

in $B(l^2(H))$ is a C*–algebra fulfulling the requirements (i) and (ii). Let us denote this C*-algebra by $\mathcal{A}$.

Assume that $\mathcal{B} \subset B(\mathcal{H})$ is another C*-algebra generated by elements $W(x)$ $(x \in H)$ satisfying (i) and (ii). We have to show an isomorphism $\alpha : \mathcal{A} \to \mathcal{B}$ such that $\alpha(R(x)) = W(x)$ $(x \in H)$. α will be constructed in several steps.

We shall need the Hilbert space

$$l^2(H,\mathcal{H}) = \left\{A : H \to \mathcal{H} : \sum_{x\in H} \|A(x)\|^2 < +\infty\right\}.$$

Set $x \otimes f$ for $x \in H$ and $f \in \mathcal{H}$ as

$$(x\otimes f)(y) = \begin{cases} f & x = y \\ 0 & x \ne y\,. \end{cases}$$

(Note that $l^2(H,\mathcal{H})$ is isomorphic to $l^2(H)\otimes\mathcal{H}$.) The application

$$y \mapsto \pi(y) \qquad \pi(y)(x\otimes f) = (x-y)\otimes W(y)f$$

is a representation of the CCR on the Hilbert space $l^2(H,\mathcal{H})$. π is equivalent to R. If a unitary $U : l^2(H,\mathcal{H}) \to l^2(H,\mathcal{H})$ is defined as

$$U(x\otimes f) = x \otimes W(x)f$$

then

$$U\,\pi(y) = (R(y)\otimes id)U \qquad (y \in H).$$

To prove our claim it is sufficient to find an isomorphism between $\mathcal{B}$ and the C*–algebra generated by $\{\pi(y) : y \in H\}$. We show that for any finite linear combination

$$\left\|\sum \lambda_i\, W(y_i)\right\| = \left\|\sum \lambda_i\, \pi(y_i)\right\| \tag{4.2}$$

holds.

Let $\hat{H}$ stand for the dual group of the discrete group H. $\hat{H}$ consists of characters of H and endowed by the topology of pointwise convergence forms a compact topological group. We consider the normalized Haar measure on $\hat{H}$. The spaces $l^2(H)$ and $L^2(\hat{H})$ are isomorphic by the Fourier transformation, which establishes the unitary equivalence between the above π and $\hat{\pi}$ defined below.

$$\hat{\pi}(y)\hat{A}(\chi) = \chi(y)W(y)\hat{A}(\chi) \qquad (y \in H\,,\ \chi \in \hat{H}\,,\ \hat{A} \in L^2(\hat{H},\mathcal{H}))$$

Hence

$$\left\|\sum \lambda_i\, \pi(y_i)\right\| = \left\|\sum \lambda_i\, \hat{\pi}(y_i)\right\|. \tag{4.3}$$

A closer look at the definition of $\hat{\pi}$ gives that $\hat{\pi}(y)$ is essentially a multiplication operator (by $\chi(y)W(y)$) and its norm is the sup norm. That is,

$$\left\|\sum \lambda_i\, \hat{\pi}(y_i)\right\| = \sup\left\{\left\|\sum \lambda_i\, \chi(y_i)W(y_i)\right\| \,:\, \chi \in \hat{H}\right\}. \tag{4.4}$$

Since the right hand side is the sup of a continuous function over $\hat{H}$, this sup may be taken over any dense set.

Let us set

$$G = \{\exp(2\mathrm{i}\,\sigma(x,\cdot) \,:\, x \in H\}.$$

Clearly, $G \subset \hat{H}$ is a subgroup. The following result is at our disposal (see (23.26) of [19]).

If $K \subset \hat{H}$ is a proper closed subgroup then there exists $0 \neq h \in H$ such that $k(h) = 1$ for every $k \in K$.

Assume that $\exp(2\mathrm{i}\,\sigma(x,y)) = 1$ for every $x \in H$. Then for every $t \in \mathbb{R}$ there exists an integer $l \in \mathbb{Z}$ such that $t\,\sigma(x,y) = l\,\pi$. This is possible if $\sigma(x,y) = 0$ (for every $x \in H$) and y must be 0. According to the above cited result of harmonic analysis the closure of G must be the whole $\hat{H}$.

Now we are in a position to complete the proof. For

$$\chi(\cdot) = \exp(2\mathrm{i}\,\sigma(x,\cdot)) \in G$$

we have

$$\begin{aligned}\left\|\sum \lambda_i\, \chi(y_i)W(y_i)\right\| &= \left\|W(x)\sum \lambda_i\, W(y_i)W(-x)\right\| = \\ &= \left\|\sum \lambda_i\, W(y_i)\right\|\end{aligned}$$

and this is the supremum in (4.4). Through (4.4) we arrive at (4.2). □

The previous theorem is due to Slawny [**Sl**]. We learnt from the proof that $\mathrm{CCR}(H,\sigma)$ has a representation on $l^2(H)$ given by (4.1). The subalgebra

$$\left\{\sum_{x\in H} \lambda(x)R(x) \,:\, \lambda \,:\, H \to \mathbb{C} \text{ has finite support}\right\}$$

is dense in $\mathrm{CCR}(H,\sigma)$ and there exists a state τ on $\mathrm{CCR}(H,\sigma)$ such that

$$\tau\left(\sum \lambda(x)R(x)\right) = \lambda(0)\,. \tag{4.5}$$

It is simple to verify that $\tau(ab) = \tau(ba)$. Therefore, τ is called the tracial state of $\mathrm{CCR}(H,\sigma)$. We can use τ to prove the following.

Proposition 4.3 *If $f, g \in H$ are different then*

$$\|W(f) - W(g)\| \geq \sqrt{2}.$$

Proof. For $h_1 \neq h_2$, we have $\tau(W(h_1)W(-h_2)) = 0$. Hence $\|W(f) - W(g)\|^2 \geq \tau((W(f) - W(g)^*(W(f) - W(g))) = 2$. □

It follows from the Proposition that the unitary group $t \mapsto W(tf)$ is never normcontinuous and the C*-algebra $\mathrm{CCR}(H, \sigma)$ can not be separable.

Slawny's theorem has also a few important consequences. Clearly for $(H_1, \sigma_1) \subset (H_2, \sigma_2)$ the inclusion $\mathrm{CCR}(H_1, \sigma_1) \subset \mathrm{CCR}(H_2, \sigma_2)$ must hold. (If H_1 is a proper subspace of H_2 then $\mathrm{CCR}(H_1, \sigma_2)$ is a proper subalgebra of $\mathrm{CCR}(H_2, \sigma_2)$.) If $T : H \to H$ is an invertible linear mapping such that

$$\sigma(f, g) = \sigma(Tf, Tg) \tag{4.6}$$

then it may be lifted into a *-automorphism of $\mathrm{CCR}(H, \sigma)$. Namely, there exists an automorphism γ_T of $\mathrm{CCR}(H, \sigma)$ such that

$$\gamma_T(W(f)) = W(Tf) \tag{4.7}$$

A simple example is the parity automorphism

$$\pi(W(f)) = W(-f) \qquad (f \in H)\,. \tag{4.8}$$

Let (H, σ) be a symplectic space. A real linear mapping $J : H \to H$ is called a complex structure if

(i) $J^2 = -I$,

(ii) $\sigma(Jf, f) \leq 0 \qquad (f \in H)$,

(iii) $\sigma(f, g) = \sigma(Jf, Jg) \qquad (f, g \in H)$.

If a complex structure J is given then H may be considered as a complex vectorspace setting

$$(t + is)f = tf + sJf \qquad (s, t \in \mathbb{R}\,,\;\; f \in H) \tag{4.9}$$

The definition

$$\langle f, g\rangle = \sigma(f, Jg) + i\,\sigma(f, g) \tag{4.10}$$

supplies us (a complex) inner product. So to have a symplectic space (over the reals) with a complex structure is equivalent to being given a complex inner product space.

Let J be a complex structure over (H, σ). The gauge automorphism

$$\gamma_\alpha(W(f)) = W(\cos\alpha\, f + J\, \sin\alpha\, f) \quad (\alpha \in [0, 2\pi]\,,\;\; f \in H) \tag{4.11}$$

is another example for lifting of a mapping into an automorphism.

We shall restrict ourselves mainly to C*-algebras associated to a nondegenerate symplectic space but degeneracy of the symplectic form appears in certain cases. Now this possibility will be discussed following the paper [25].

Let σ be (a possible degenerate) symplectic form on H. We write $\Delta(H, \sigma)$ for the free vectorspace generated by the symbols $\{W(h) : h \in H\}$. So $\Delta(H, \sigma)$ consists of formal finite linear combinations like

$$\sum \lambda_i\, W(h_i)\,.$$

We may endow $\Delta(H,\sigma)$ by a *-algebra structure by setting

$$W(h)^* = W(-h) \qquad (h \in H) \tag{4.12}$$

and

$$W(h)W(g) = \exp(\mathrm{i}\,\sigma(x,y))W(h+y) \qquad (h,y \in H) \tag{4.13}$$

On the *-algebra $\Delta(H,\sigma)$ we shall consider the so-called minimal regular norm (cf. [29], Ch. IV §18.3). We take all *-representations π of $\Delta(H,\sigma)$ by bounded Hilbert space operators and define

$$\|a\| = \sup\{\|\pi(a)\| \,:\, \pi \text{ is a representation}\} \qquad (a \in \Delta(H,\sigma)) \tag{4.14}$$

Another possibility is to take all positive normalized functionals (that is, states) φ on $\Delta(H,\sigma)$ and to introduce the norm

$$\|a\| = \sup\{\varphi(a^*a)^{1/2} \,:\, \varphi \text{ is a state}\} \qquad (a \in \Delta(H,\sigma)) \tag{4.15}$$

One can see that (4.14) and (4.15) determine the same norm, called the minimal regular norm. The completion of $\Delta(H,\sigma)$ with respect to $\|\cdot\|$ will be a C*–algebra and it is CCR(H,σ) by definition. It follows from Slawny's theorem that for nondegenerate σ the previous and the latter definitions coincide.

Now we study the extreme case when $\sigma \equiv 0$. Then $\Delta(H,\sigma)$ is commutative and a state φ of it corresponds to a positive-definite function F on the discrete abelian group H. We have

$$\varphi\Big(\sum \overline{\lambda_i}\, W(h_i)^* \sum \lambda_j\, W(h_j)\Big) \geq 0$$

for every $\lambda_i \in \mathbb{C}$ and $h_i \in H$ if and only if the function

$$F \,:\, h \mapsto \varphi(W(h)) \qquad (h \in H)$$

is positive-definite. Due to Bochner's theorem ([19], 33.1) there is a probability measure μ on the compact dual group $\hat{H}$ such that

$$F(h) = \int \chi(h)\, d\mu(\chi) \qquad (h \in H)\,.$$

Hence

$$\sup\left\{\varphi(a^*a)^{1/2} \,:\, \varphi \text{ is a state}\right\} = \sup\left\{\chi(a^*a)^{1/2} \,:\, \chi \in \hat{H}\right\},$$

where for $a = \sum \lambda_i\, W(h_i) \in \Delta(H,\sigma)$ $\quad \chi(a)$ (or $a(\chi)$) is defined as

$$\sum \lambda_i\, \chi(h_i)\,.$$

In this way every element a of $\Delta(H,\sigma)$ may be viewed to be a continuous function on $\hat{H}$ and

$$\|a\| = \sup\left\{|a(\chi)| \,:\, \chi \in \hat{H}\right\} \qquad (a \in \Delta(H,\sigma))\,.$$

$\Delta(H,\sigma)$ evidently separates the points of $\hat{H}$ and the Stone-Weierstrass theorem tells us that CCR(H,σ) is isomorphic to the C*-algebra of all continuous functions on the compact space $\hat{H}$.

The case of a vanishing symplectic form does not occur frequently, however, it may happen that $H = H_0 \oplus H_1$ and

$$\sigma\,(h_0 \oplus h_1\;,\;h_0' \oplus h_1') = \sigma_1\,(h_1 \oplus h_1')$$

with a nondegenerate symplectic form σ_1 on H_1. Then the *-algebra $\Delta(H,\sigma)$ is the algebraic tensor product of $\Delta(H_0, 0)$ and $\Delta(H_1, \sigma_1)$ and $\mathrm{CCR}(H,\sigma)$ will be

$$\mathrm{CCR}(H_0, 0) \otimes \mathrm{CCR}(H_1, \sigma_1)\,. \tag{4.16}$$

(Note that since $\mathrm{CCR}(H_0, 0)$ is commutative, the C*-norm on the tensor product is unique.)

Now we review briefly the general case. For a degenerate symplectic form σ we set

$$H_0 = \{x \in H \;:\quad \sigma(x,y) = 0 \quad \text{for every} \quad y \in H\}$$

for the kernel of σ. $\Delta(H_0, 0)$ is the center of the *-algebra $\Delta(H,\sigma)$ and there exists a natural projection E given by

$$E\left(\sum_{x\in H} \lambda(x) W(x)\right) = \sum_{x\in H_0} \lambda(x) W(x) \tag{4.17}$$

and mapping $\Delta(H,\sigma)$ onto $\Delta(H_0, 0)$. Having introduced the minimal regular norm we observe that $\mathrm{CCR}(H_0, 0)$ is the center of $\mathrm{CCR}(H,\sigma)$ and E is a conditional expectation. The maximal *-ideals of $\mathrm{CCR}(H,\sigma)$ are in one-to-one correspondence with those of $\mathrm{CCR}(H_0, 0)$. In particular, $\mathrm{CCR}(H,\sigma)$ is simple if and only if $H_0 = \{0\}$, that is, σ is nondegenerate. Concerning the details we refer to [25].

For a nondegenerate symplectic form Slawny's theorem provides readily that $\mathrm{CCR}(H,\sigma)$ is simple.

Chapter 5

Fock representation

5.1 The Fock state

Let $\mathcal{H}$ be a Hilbert space and CCR($\mathcal{H}$) be the corresponding CCR-algebra.

Theorem 5.1 *There is a state φ on the C^*-algebra CCR($\mathcal{H}$) such that*

$$\varphi(W(f)) = \exp(-\|f\|^2/2). \tag{5.1}$$

Proof. $\varphi(I) = 1$ follows from $f = 0$. The state φ exists if $\varphi(A^*A) \geq 0$ when A is a linear combination of Weyl operators. Assume that $A = \sum_i \lambda_i W(f_i)$. Then

$$\begin{aligned} A^*A &= \sum_i \overline{\lambda_i} W(-f_i) \sum_j \lambda_j W(f_j) \\ &= \sum_{i,j} \overline{\lambda_i}\lambda_j W(f_j - f_i) \exp \tfrac{1}{2}(-\langle f_i, f_j\rangle + \langle f_j, f_i\rangle) \end{aligned}$$

and

$$\begin{aligned} \varphi(A^*A) &= \sum_{i,j} \overline{\lambda_i}\lambda_j \exp \tfrac{1}{2}(-\langle f_j - f_i, f_j - f_i\rangle - \langle f_i, f_j\rangle + \langle f_j, f_i\rangle) \\ &= \sum_{i,j} \overline{\lambda_i}\lambda_j \exp \tfrac{1}{2}(-\|f_j\|^2 - \|f_i\|^2) \exp 2\langle f_j, f_i\rangle). \end{aligned}$$

This is positive (for all λ_i) if the matrx

$$(i, j) \mapsto \exp \tfrac{1}{2}(-\|f_j\|^2 - \|f_i\|^2) \exp 2\langle f_j, f_i\rangle)$$

is positive. This is the entry-wise product of the matrices

$$(i, j) \mapsto \exp \tfrac{1}{2}(-\|f_j\|^2 - \|f_i\|^2) \text{ and } (i, j) \mapsto \exp 2\langle f_j, f_i\rangle).$$

Due to the Hadamard theorem, it is enough to see that both are positive. The first one has the form X^*X, so it is positive. The second one is the entry-wise exponential of the positive Gram matrix $(\langle f_j, f_i\rangle)_{ij}$. This is positive as well.

The linear functional φ is defined on the linear combinations of the Weyl operators and it is positive. By continuity, it can be extended to the whole CCR($\mathcal{H}$). $\square$

The state defined by (5.1) is called **Fock state**.

Next we perform the GNS-representation. $\mathcal{H}_\varphi$ is the Hilbert space generated by $\mathrm{CCR}(\mathcal{H})$ with the inner product $\langle A, B\rangle := \varphi(A^*B)$. The vector I is usually denoted by Φ and called **vacuum vector**. The representation $\pi_\varphi : \mathrm{CCR}(\mathcal{H}) \to B(\mathcal{H}_\varphi)$ is defined as

$$\pi_\varphi(B)A = BA \qquad (A, B \in \mathrm{CCR}(\mathcal{H})).$$

The represntation π_φ is the **Fock representation**.

The example in Chapter 1 corresponds to the Fock representation of $\mathrm{CCR}(\mathcal{H})$ when $\mathcal{H}$ has dimension 1. Lemma 1.4 gives the example

$$W(x + \mathrm{i}y) = \exp \mathrm{i}\sqrt{2}(xP + yQ).$$

The formula (1.20) shows that φ_0 is the vacuum vector and the Hilbert space $\mathcal{H}_\varphi$ is $L^2(\mathbb{R})$.

In the rest of this chapter $W(f)$ and $\pi_\varphi(W(f))$ will be identified.

5.2 Field operators

An important property of this representation that the one-parameter group $U_t := W(tf)$ of unitaries is weakly continuous, since the function

$$t \mapsto \langle W(g), W(tf)W(h)\rangle$$

is continuous for every $g, h \in \mathcal{H}$. Due to the Stone theorem, there is a self-adjoint operator $B(f)$ on $\mathcal{H}_\varphi$ such that

$$W(tf) = \exp(\mathrm{i}tB(f)) \qquad (t \in \mathbb{R}). \tag{5.2}$$

It follows from Proposition 4.3 that the **field operator** $B(f)$ must be unbounded. The vectors $W(g)\Phi$ are in the domain of $B(f)$ and more generally, in the domain of $B(f_1)B(f_2)\dots B(f_k)$. The expression $\varphi(B(f_1)B(f_2)\dots B(f_k))$ is defined as

$$\langle \Phi, B(f_1)B(f_2)\dots B(f_k)\Phi\rangle.$$

Proposition 5.2 *Then for $f, g \in \mathcal{H}$ and $t \in \mathbb{R}$ the following relations hold in the Fock representation.*

(i) $B(tf) = tB(f), \quad B(f + g) = B(f) + B(g)$.

(ii) $[B(f), W(g)] = 2\sigma(f, g)W(g), \quad [B(f), B(g)] = -2i\sigma(f, g)$.

(iii) $\varphi(B(f)B(g)) = \langle B(f)\Phi, B(g)\Phi\rangle = \langle f, g\rangle$.

Set

$$B^{\pm}(f) = \frac{1}{2}(B(f) \mp \mathrm{i}B(\mathrm{i}f))\,. \tag{5.3}$$

Then

$$B(f) = B^+(f) + B^-(f)) \tag{5.4}$$

and

$$[B^-(f), B^+(g)] = \langle g, f\rangle \qquad (f, g \in \mathcal{H})\,. \tag{5.5}$$

$B^+(f)$ is called **creation operator** and $B^-(f)$ is **annihilation operator**.

Lemma 5.3 $B^-(f)\Phi = 0$.

Lemma 5.4 *For $k \in \mathbb{Z}$ we have*

$$B^-(f)B^+(f)^k\Phi = k\|f\|^2 B^+(f)^{k-1}\Phi \qquad (f \in H)\,.$$

Proof. We apply induction. The case $k = 0$ is contained in the previous lemma. Due to the commutation relation (5.5) we have

$$\begin{aligned} B^-(f)B^+(f)^{k+1}\Phi &= (B^+(f)B^-(f) + \langle f, f\rangle)B^+(f)^k\Phi \\ &= (k-1)\|f\|^2B^+(f)^k\Phi + \|f\|^2B^+(f)^k\Phi\,. \end{aligned}$$

□

One obtains by induction again the following.

Proposition 5.5 *If $n, k \in \mathbb{N}$ and $f \in \mathcal{H}$, then*

$$B^-(f)^nB^+(f)^k\Phi = \begin{cases} 0 & \text{if} \quad n > k \\ \frac{k!}{(k-n)!}\|f\|^{2n}B^+(f)^{k-n}\Phi & \text{if} \quad n \le k\,. \end{cases}$$

Example 5.6 We assume that $\mathcal{H}$ is of one dimension. Fix a unit (basis) vector η in $\mathcal{H}$ and set

$$f_n = \frac{1}{\sqrt{n!}}B^+(\eta)^n\Phi \qquad (n \in \mathbb{Z}_+)\,. \tag{5.6}$$

Then $\{f_0, f_1, \dots\}$ is an orthonormal basis in the Fock space. If we write a^+ for $B^+(\eta)$ and a for $B^-(\eta)$ then

$$a^+f_n = \sqrt{n+1}f_{n+1} \qquad af_n = \begin{cases} \sqrt{n}f_{n-1} & n \ge 1 \\ 0 & n = 0 \end{cases}$$

and

$$[a, a^+] = 1\,.$$

With the choice

$$q = \frac{1}{\sqrt{2}}(a + a^+) \quad \text{and} \quad p = \frac{i}{\sqrt{2}}(a^+ - a)$$

the Heisenberg commutation relation is satisfied.

The vector f_n is called n-particle vector in the physics literature. Transforming f_n into f_{n+1} the operator a^+ increases the number of particles. This is the origin of the term **creation operator**. The operator a **annihilates** in the similar sense.

Our present formulas are very similar to those in Chapter 1. The Fock space $\mathcal{K}$ can be identified with $L^2(\mathbb{R})$ if the vector f_n corresponds to the Hermite function $\varphi_n \in L^2(\mathbb{R})$. It is clear that the operators a and a^+ are the same.

We compute the coordinates of the vectors $W(z)\Phi$ in the basis $\{f_n : n \in \mathbb{Z}^+\}$. For the sake of simplicity we choose $\eta = 1$.

$$\langle W(z)\Phi, f_n\rangle \;=\; \langle \exp(iB^+(z) + iB^-(z))\Phi, f_n\rangle$$

$$
\begin{aligned}
&= \exp\left(-\frac{1}{2}[iB^+(z), iB^-(z)]\right)\langle \exp(iB^+(z))\Phi, f_n\rangle \\
&= \exp\left(-\frac{1}{2}|z|^2\right)\sum \frac{(iz)^m}{m!}\langle \sqrt{m!} f_m, f_n\rangle \\
&= \exp\left(-\frac{1}{2}|z|^2\right)\frac{(iz)^n}{\sqrt{n!}}
\end{aligned}
$$

Hence for any $z \in \mathbb{C}$ the associated exponential vector $e(z)$ is the sequence

$$
\left(1, iz, \frac{(iz)^2}{\sqrt{z!}}, \dots, \frac{(iz)^n}{\sqrt{n!}}, \cdots\right) \equiv \sum_n \frac{(iz)^n}{\sqrt{n!}} f_n. \tag{5.7}
$$

□

5.3 Fock space

Let $\{\eta_i : i \in I\}$ be an orthonormal basis in the complex Hilbert space $\mathcal{H}$. We set

$$
\left|\eta_{i_1}^{n_1}; \eta_{i_2}^{n_2}; \dots; \eta_{i_k}^{n_k}\right\rangle = \frac{1}{\sqrt{n_1! \dots n_k!}} B^+(f_{i_1})^{n_1} \dots B^+(f_{i_k})^{n_k}\Phi\,. \tag{5.8}
$$

So for every choice of different indices $i_1, i_2, \dots, i_k$ in I and $n_1, n_2, \dots, n_k \in \mathbb{N}$ we get to a unit vector in $\mathcal{K}$. The vectors

$$
\left|\eta_{i_1}^{n_1}; \dots, \eta_{i_k}^{n_k}\right\rangle \quad \text{and} \quad \left|\eta_{j_1}^{m_1}; \dots, \eta_{j_l}^{m_l}\right\rangle
$$

are different if $((n_1, i_1), \dots, (n_k, i_k))$ is not a permutation of $((m_1, j_1), \dots, (m_l, j_l))$ and in this case they are orthogonal. All such vectors form a canonical orthonormal basis in $\mathcal{K}$.

From $\varphi(B(f)B(g)) = \langle f, g\rangle$ we deduce

$$
\varphi(B^\pm(f)B^\pm(g)) = 0
$$

if $f \perp g$. If the sequence $f_1, f_2, \dots, f_n$ in $\mathcal{H}$ has the property that any two vectors are orthogonal or identical then in the expansion (5.22) of

$$
\varphi(B^\pm(f_n)B^\pm(f_{n-1}) \dots B^\pm(f_1))
$$

we may have a nonzero term if always identical vectors are paired together. We benefit from this observation in the next proposition.

Proposition 5.7 *Assume that $g_1, g_2, \dots, g_k$ are pairwise orthogonal vectors in H. Then*

$$
B^+(g_1)^{m_1} B^+(g_2)^{m_2} \dots B^+(g_k)^{m_k}\Phi
$$

and

$$
B^+(g_1)^{n_1} B^+(g_2)^{n_2} \dots B^+(g_k)^{n_k}\Phi
$$

are orthogonal whenever $m_j \neq n_j$ for at least one $1 \leq j \leq k$.

Proof. Suppose that $m_1 \neq n_1$ and $n_1 > m_1$. The inner product of the above vectors is given by

$$\varphi\left(B^-(g_k)^{n_k}\dots B^-(g_1)^{n_1}B^+(g_1)^{m_1}\dots B^+(g_k)^{m_k}\right)$$

and equals to

$$\varphi\left(B^-(g_1)^{n_1}B^+(g_2)^{m_1}\right)\varphi\left(B^-(g_k)^{n_k}\dots B^-(g_2)^{n_2}B^+(g_2)^{m_2}\dots B^+(g_k)^{m_k}\right).$$

Here the first factor vanishes due to $n_1 > m_1$.

We are able to conclude also the formula

$$\|B^+(g_1)^{n_1}B^+(g_2)^{n_2}\dots B^+(g_k)^{n_k}\Phi\|^2 = n_1!n_2!\dots n_k! \tag{5.9}$$

provided that $\|g_1\| = \|g_2\| = \dots = \|g_k\| = 1$.

Lemma 5.8 *For $g_1, g_2, \dots, g_n, f \in H$ with $\|f\| = 1$ we have*

$$\|B(f)B(g_1)B(g_2)\dots B(g_n)\Phi\| \le 2\sqrt{n+1}\|B(g_1)\dots B(g_n)\Phi\|.$$

We consider the linear subspace spanned by the vectors $f, g_1, g_2, \dots, g_n$ and take an orthonormal basis $f_1 = f, f_2, \dots, f_k$. We may express $B(g_i)$ by creation and annihilation operators corresponding to the basis vectors and get

$$\eta = B(g_1)\dots B(g_n)\Phi = \sum \lambda(n_1,\dots,n_k)B^+(f_1)^{n_1}\dots B^+(f_k)^{n_k}\Phi$$

(Here the summation is over the k-triples $(n_1,\dots,n_k)$ such that $n_i \in \mathbb{Z}_+$ and $\sum n_i \le n$.) Since

$$\|B(f)\eta\| \le \|B^+(f_1)\eta\| + \|B^-(f_1)\eta\|$$

it suffices to show that

$$\|B^\pm(f_1)\eta\|^2 \le (n+1)\|\eta\|^2.$$

Now we estimate as follows.

$$\begin{aligned}
\|B^+(f_1)\eta\|^2 &= \|\sum \lambda(n_1,\dots,n_k)B^+(f_1)^{n_1+1}B^+(f_2)^{n_2}\dots B^+(f_k)^{n_k}\Phi\|^2\\
&= \sum \|\lambda(n_1,\dots,n_k)B^+(f_1)^{n_1+1}B^+(f_2)^{n_2}\dots B^+(f_k)^{n_k}\Phi\|^2\\
&= \sum (n_1+1)\|\lambda(n_1,\dots,n_k)B^+(f_1)^{n_1}\dots B^+(f_k)^{n_k}\Phi\|^2\\
&\le (n+1)\|B(g_1)\dots B(g_n)\Phi\|^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|B^-(f_1)\eta\|^2 &= \|\sum_{n_1\ge 1} \lambda(n_1,\dots,n_k)n_1B^+(f_1)^{n_1-1}B^+(f_2)^{n_2}\dots B^+(f_k)^{n_k}\Phi\|^2\\
&= \sum n_1^2|\lambda(n_1,\dots,n_k)|^2\|B^+(f_1)^{n_1-1}B^+(f_2)^{n_2}\dots B^+(f_k)^{n_k}\Phi\|^2\\
&= \sum n_1\|\lambda(n_1,\dots,n_k)B^+(f_1)^{n_1}B^+(f_2)^{n_2}\dots B^+(f_k)^{n_k}\Phi\|^2\\
&\le n\sum \|\lambda(n_1,\dots,n_k)B^+(f_1)^{n_1}B^+(f_2)^{n_2}\dots B^+(f_k)^{n_k}\Phi\|^2 = n\|\eta\|^2.
\end{aligned}$$

Lemma 5.4 and the explicit norm expression (5.9) have been used.

Let the Fock representation act on a Hilbert space $\mathcal{H}$ containing the vacuum vector Φ. The linear span of the vectors $B(g_1)B(g_2)\dots B(g_n)\Phi$ $(g_1, g_2, \dots, g_n \in H,\ n \in \mathbb{N})$ and $B^+(g_1)B^+(g_2)\dots B^+(g_n)\Phi$ $(g_1, g_2, \dots,\ g_n \in H,\ n \in \mathbb{N})$ coincide. It will be denoted by $\mathcal{D}_B$. So far it is not clear whether $\mathcal{D}_B$ is complete. This is what we are going to show.

Let A be a linear operator on a Hilbert space $\mathcal{K}$. A vector $\xi \in \mathcal{K}$ is called **entire analytic** (for A) if ξ is in the domain of A^n for every $n \in \mathbb{N}$ and

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \|A^k \xi\| < +\infty$$

for every $t > 0$. If ξ is an entire analytic vector then $\exp(zA)\xi$ makes sense for every $z \in \mathbb{C}$ and it is an entire analytic function of z.

Theorem 5.9 *$\mathcal{D}_B$ consists of entire analytic vectors for $B(f)$ $(f \in H)$.*

Proof. Let $\xi = B(f_1)B(f_2)\dots B(f_n)\Phi \in \mathcal{D}_B$. By a repeated application of Lemma 5.8 we have

$$\|B(f)^k \xi\| \le 2^k \sqrt{\frac{(n+k)!}{n!}} \|\xi\|$$

and it is straightforward to check that the power series

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \|B(f)^k \xi\|$$

converges for every t. Since the entire analytic vectors form a linear subspace, the proof is complete. □

Due to Theorem 5.9 every vector $W(f)\Phi = \exp(iB(f))\Phi$ can be approximated (through the power series expansion of the exponential function) by elements of $\mathcal{D}_B$. This yields, immediately that $\mathcal{D}_B$ is dense in $\mathcal{H}$. According to Nelson's theorem on analytic vectors (see [34] X.39), $\mathcal{D}_B$ is a core for $B(f)$ $(f \in H)$, in other words, $B(f)$ is the closure of its restriction to $\mathcal{D}_B$. It follows also that $\mathcal{D}_B$ is core for $B^{\pm}(f)$ and $B^-(f)^* = B^+(f)$.

Theorem 5.10 *The Fock representation is irreducible.*

Proof. We have to show that for any $0 \ne \eta \in \mathcal{H}$ the closed linear subspace $\mathcal{H}_1$ generated by $\{W(f)\eta : f \in H\}$ is $\mathcal{H}$ itself. Let $\mathcal{M}$ be the von Neumann algebra generated by the unitaries $\{W(f) : f \in H\}$ in $B(\mathcal{H})$. Clearly, $\mathcal{M}\eta \subset \mathcal{H}_1$.

We consider a canonical basis in $\mathcal{H}$ consisting of vectors (5.8). $\eta \in \mathcal{H}$ has an expansion as (countable) linear combinations of basis vectors. Assume that a vector

$$|f_1^{n_1}; f_2^{n_2}; \dots; f_k^{n_k}\rangle \tag{5.10}$$

has a nonzero coefficient.

The operator

$$B^+(f_1)B^-(f_1)\dots B^+(f_k)B^-(f_k) \tag{5.11}$$

is selfadjoint and (4.5) is its eigenvector with eigenvalue $n_1 + n_2 + \ldots + n_k$. Since (5.11) is affiliated with $\mathcal{M}$, its spectral projections are in $\mathcal{M}$. In this way we conclude that the vector (5.10) lies in $\mathcal{H}_1$.

It is easy to see that

$$B^{\pm}(f)\mathcal{H}_1 \subset \mathcal{H}_1$$

for every $f \in H$. By application of the annihilation operators $B^-(f_i)$ $(1 \le i \le k)$ we obtain that the cyclic (vacuum) vector Φ is in $\mathcal{H}_1$. Therefore, $\mathcal{H}_1 = \mathcal{H}$ must hold. □

Next we introduce some vectors of special importance by means of the Weyl operators. For $f \in H$ set

$$e(f) = \exp\left(\tfrac{1}{2}\|f\|^2\right) W(f)\Phi\,, \tag{5.12}$$

which is called **exponential vector**. One may compute that

$$\langle e(f), e(g)\rangle = \exp\langle g, f\rangle \qquad (f, g \in H) \tag{5.13}$$

Proposition 5.11 $\{e(f) : f \in H\}$ *is a linearly independent complete subset of* $\mathcal{H}$.

Proof. We use the fact that the family $\{e^{tx} : x \in \mathbb{R}\}$ of exponential functions is linearly independent.

Let $f_1, f_2, \ldots, f_n \in H$ be a sequence of different vectors and assume that $\sum \lambda_i e(f_i) = 0$. We choose a vector $g \in H$ such that the numbers

$$\mu_i = \langle f_i, g\rangle \quad (1 \le i \le n)$$

are distinct. For any $t \in \mathbb{R}$ we have

$$0 = \langle e(tg), \sum \lambda_i e(f_i)\rangle = \sum \lambda_i \exp(t\langle f_i, g\rangle)$$

and we may conclude that $\lambda_i = 0$ for every $1 \le i \le n$.

Due to the cyclicity of the vacuum vector Φ the set $\{e(f) : f \in H\}$ is complete. A little bit more is true. The norm expression

$$\|e(f) - e(g)\|^2 = \exp(\|f\|^2) + \exp\|g\|^2 - 2\mathrm{Re}\ \exp\langle f, g\rangle \tag{5.14}$$

tells us that the mapping $f \mapsto e(f)$ is norm continuous. Therefore $\{e(f) : f \in S\}$ is complete whenever S is a dense subset of $\mathcal{H}$. □

Example 5.12 Assume that $\mathcal{H}$ has two dimension with orthogonal unit vectors η_1 and η_2. We use the notation $B^+(\eta_i) = B_i^+$ and $B^-(\eta_i) = B_i^-$ $(i = 1, 2)$. The vectors

$$f_i^{(1)} = B_i^+\Phi \qquad (i = 1, 2)$$

are orthonormal and orthogonal to Φ:

$$\langle \Phi, f_i^{(1)}\rangle = \langle B_i^-\Phi, \Phi\rangle = 0 \quad \text{due to Lemma 5.3},$$

$$\langle f_i^{(1)}, f_j^{(1)}\rangle = \langle B_i^+\Phi, B_j^+\Phi\rangle = \langle B_i\Phi, B_j\Phi\rangle = \langle \eta_i, \eta_j\rangle \quad \text{due to (iii) in Proposition 5.2.}$$

In the previous formalism it was

$$f_i^{(1)} = |\eta_i\rangle \qquad (i = 1, 2).$$

The next subspace has 3 dimension:

$$|\eta_1^2\rangle = \frac{1}{\sqrt{2!}}(B_1^+)^2\Phi, \quad |\eta_2^2\rangle = \frac{1}{\sqrt{2!}}(B_2^+)^2\Phi, \quad |\eta_1, \eta_2\rangle = B_1^+ B_2^+ \Phi.$$

The subspaces of the Fock space are antisymmetric tensor powers of the Hilbert space $\mathcal{H}$ which is two dimensional now. Thext subspace is the third power of $\mathcal{H}$:

$$|\eta_1^3\rangle = \frac{1}{\sqrt{3!}}(B_1^+)^3\Phi, \quad |\eta_1^2, \eta_2\rangle = \frac{1}{\sqrt{2!}}(B_1^+)^2 B_2^+\Phi,$$

$$|\eta_1, \eta_2^2\rangle = \frac{1}{\sqrt{2!}}B_1^+(B_2^+)^2\Phi, \quad |\eta_2^3\rangle = \frac{1}{\sqrt{3!}}(B_2^+)^3\Phi.$$

The situation can be continued. $\square$

5.4 The positivity condition

To determine a state on CCR(H, σ), it is enough to give the values on the unitaries $W(f)$ $(f \in \mathcal{H})$. When the Fock state was introduced, an argument was required to show the existence. This will be extended now.

Let X be an arbitrary (nonempty) set. A function $F : X \times X \to \mathbb{C}$ is called a **positive definite kernel** if and only if

$$\sum_{j,k=1}^{n} c_j \, \overline{c}_k \, F(x_j, x_k) \geq 0$$

for all $n \in \mathbb{N}$, $\{x_1, x_2, \ldots, x_n\} \subset X$ and $\{c_1, c_2, \ldots, c_n\} \subset \mathbb{C}$. The product of positive definite kernels is positive definite. This statement is equivalent to the **Hadamard theorem** which says that the entry-wise product of positive matrices is positive.

Proposition 5.13 *Let (H, σ) be a symplectic space and $G : H \to \mathbb{C}$ a function. There exists a state φ on CCR(H, σ) such that*

$$\varphi(W(f)) = G(f) \qquad (f \in H)$$

if and only if $G(0) = 1$ and the kernel

$$(f, g) \mapsto G(f - g) \exp(-\mathrm{i}\,\sigma(f, g))$$

is positive definite.

Proof. For $x = \sum c_j\, W(f_j)$ we have

$$xx^* = \sum_{j,k} c_j\, \overline{c}_k\, W(f_j - f_k) e^{-i\sigma(f_j,f_k)}\,.$$

Since $\varphi(xx^*) \geq 0$ should hold, we see that the positivity condition is necessary.

On the other hand, the positivity condition allows us to define a positive functional on the linear hull of the Weyl operators and a continuous extension to $\mathrm{CCR}(H, \sigma)$ supplies a state. □

Lemma 5.14 *Let (H, σ) be a symplectic space. (It might be degenerate.) If $\alpha(\,\cdot\,,\,\cdot\,)$ is a positive symmetric bilinear form on H then the following conditions are equivalent.*

(i) The kernel $(f, g) \mapsto \alpha(f, g) - \mathrm{i}\,\sigma(f, g)$ is positive definite.

(ii) $\alpha(z, z)\, \alpha(x, x) \geq \sigma(z, x)^2$ for every $x, z \in H$.

Proof. Both condition (i) and (ii) hold on H if and only if they hold on all finite dimensional subspaces. Hence we may assume that H is of finite dimension.

If $\alpha(x, x) = 0$ then both condition (i) and (ii) imply that $\sigma(x, y) = 0$ for every $y \in H$. Due to possible factorization we may assume that α is strictly positive and it will be viewed as an inner product.

There is an operator Q such that

$$\sigma(x, y) = \alpha(Qx, y) \qquad (x, y \in H)\,.$$

and $Q^* = -Q$ follows from $\sigma(x, y) = -\sigma(y, x)$. According to linear algebra in a certain basis the matrix of Q has a diagonal form $\mathrm{Diag}\,(A_1, A_2, \ldots, A_k)$, where A_i is a 1×1 0-matrix or

$$A_i = \begin{pmatrix} 0 & a_i \\ -a_i & 0 \end{pmatrix}.$$

(The first possibility occurs only if σ is degenerate.) It is easy to see that condition (i) is equivalent to $|a_i| \leq 1$ and so is condition (ii). □

Theorem 5.15 *Let (H, σ) be a symplectic space and $\alpha : H \times H \to \mathbb{R}$ a symmetric positive bilinear form such that*

$$\sigma(f, g)^2 \leq \alpha(f, f)\alpha(g, g) \qquad (f, g \in H)\,. \tag{5.15}$$

Then there exists a state φ on $CCR(H, \sigma)$ such that

$$\varphi(W(f)) = \exp\left(-\tfrac{1}{2}\alpha(f, f)\right) \qquad (f \in H)\,. \tag{5.16}$$

Proof. We are going to apply Proposition 5.13. Due to the positivity condition

$$\begin{aligned}
&\sum_{j,k} c_j \overline{c_k} \exp\left(-\tfrac{1}{2}\alpha(f_j - f_k, f_j - f_k) - \mathrm{i}\sigma(f_j, f_k)\right) \\
&\quad = \sum_{j,k} \left(c_j \exp\left(-\tfrac{1}{2}\alpha(f_j, f_j)\right)\right) \left(\overline{c}_k \exp\left(-\tfrac{1}{2}\alpha(f_k, f_k)\right)\right) \\
&\qquad\qquad \times \exp(\alpha(f_j, f_k) - \mathrm{i}\sigma(f_j, f_k)) \\
&\quad = \sum_{j,k} b_j \overline{b}_k \exp(\alpha(f_j, f_k) - i\sigma(f_j, f_k))
\end{aligned}$$

should be shown to be nonnegative. According to Lemma 5.14

$$(\alpha(f_j, f_k) - i\sigma(f_j, f_k))_{j;k}$$

is positive definite and entrywise exponentiation preserves this property. □

A state φ on CCR(H, σ) determined in the form (5.16) is called **quasifree**.

A state is **regular** if in the GNS-representation the field operators exist.

Proposition 5.16 *Let φ be a state on CCR(H, σ). If*

$$\lim_{t \to 0} \varphi(W(tf)) = 1 \qquad (f \in H),$$

then φ is regular.

Proof. We set $G(f) = \varphi(W(f))$ $(f \in H)$. According to Proposition 5.13 the matrix

$$\begin{bmatrix} 1 & G(-f_1) & G(-f_2) \\ G(f_1) & 1 & G(f_1 - f_2)\exp(-i\sigma(f_1, f_2)) \\ G(f_2) & G(f_2 - f_1)\exp(i\sigma(f_1, f_2)) & 1 \end{bmatrix}$$

is positive definite. From this we obtain

$$|G(f_2) - G(f_1)| \le 4|1 - G(f_2 - f_1)\exp(-i\sigma(f_2, f_1))|\,. \tag{5.17}$$

Combining (5.17) with the hypothesis we arrive at the continuity of the function

$$t \mapsto G(tf + g) \qquad (t \in \mathbb{R})$$

for every $f, g \in H$. Let $(\pi_\varphi, \mathcal{H}_\varphi, \Phi)$ stand for the GNS-triple. We verify by computation the continuity of the function

$$t \mapsto \langle \pi_\varphi(W(tf))\pi_\varphi(W(g_1))\Phi, \pi_\varphi(W(g_2))\Phi \rangle \qquad (t \in \mathbb{R})$$

and the regularity of φ is proven. □

It folows that a quasifree state is regular.

5.5 Analytic states

A state φ on CCR(H, σ) is said to be **analytic** if the numerical function

$$t \mapsto \varphi(W(tf)) \qquad (t \in \mathbb{R})$$

is analytic. Quasifree states are obviously analytic.

Assume that π is a regular representation of CCR(H, σ). The **field operator** $B(g)$ is obtained by differentiation of the function

$$t \mapsto \pi(W(tg))\eta = \exp(itB(g))\eta \qquad (t \in \mathbb{R})\,. \tag{5.18}$$

More precisely, if (5.18) is weakly differentiable at $t = 0$ and the derivative is $\xi \in \mathcal{H}_\varphi$, then η is in the domain of $B(g)$ and

$$\mathrm{i}B(g)\eta = \xi\,.$$

Proposition 5.17 *Let φ be an analytic state on CCR(H,σ) with GNS-triple $(\pi_\varphi, \mathcal{H}_\varphi, \Phi)$. Then $\pi_\varphi(W(g))\Phi$ is in the domain of*

$$B(f_n)B(f_{n-1})\dots B(f_1)$$

for every $g, f_1, f_2, \dots, f_n \in H$ and $n \in \mathbb{N}$.

Proof. We apply induction and suppose that

$$\eta = B(f_{n-1})\dots B(f_1)\pi_\varphi(W(g))\Phi$$

makes sense. For the sake of simpler notation we omit π_φ in the proof.

It suffices to show that

$$\lim_{t\to 0} t^{-1}\langle (W(f_n)-I)\eta, \xi\rangle = F(\xi) \tag{5.19}$$

exists if ξ is in a dense subset $\mathcal{D}_0$ of $\mathcal{H}_\varphi$ and $|F(\xi)| \le C\|\xi\|$. This ensures that

$$t \mapsto W(tf_n)\eta$$

is differentiable in the weak sense. Since for $\xi = W(h)\Phi$ the limit in (5.19) equals to

$$(-\mathrm{i})^n \frac{\partial}{\partial t}\frac{\partial^{n-1}}{\partial t_{n-1}\partial t_{n-2}\dots\partial t_1}\varphi(W(-h)W(tf_n)\dots W(t_1f_1)W(g))$$

at the point $t = t_{n-1} = t_{n-2} = \dots = t_1 = 0$, the function F is defined on the linear hull $\mathcal{D}_W$ of the vectors

$$\{W(h)\Phi : h \in H\}.$$

F is linear on $\mathcal{D}_W$ and by differentiation one can see that

$$C = \lim_{t\to 0}\frac{1}{t}\|(W(tf_n)-I)\eta\|$$

exists and it fulfils $|F(\xi)| \le C\|\xi\|$ for $\xi \in \mathcal{D}_W$. □

Although $B(f) \notin \mathrm{CCR}(H,\sigma)$, it will be rather convenient to write $\varphi(B(f_n)B(f_{n-1})\dots B(f_1))$ instead of $\langle B(f_n)B(f_{n-1})\dots B(f_1)\Phi, \Phi\rangle$. We shall keep also the notation $\mathcal{D}_W$ from the above proof. Remember that $\mathcal{D}_W$ as well as the superset Hilbert space $\mathcal{H}_\varphi$ depend on the state φ even if it is excluded from the notation.

Proposition 5.18 *Let φ be an analytic state on CCR(H,σ). Then for $f, g \in H$ and $t \in \mathbb{R}$ the following relations hold on $\mathcal{D}_W$.*

(i) $B(tf) = tB(f), \quad B(f+g) = B(f) + B(g).$

(ii) $[B(f), W(g)] = 2\sigma(f,g)W(g), \quad [B(f), B(g)] = -2i\sigma(f,g).$

Proof. The relations are deduced by derivation from (the Weyl form of) the CCR. □

5.6 Quasifree states

Recall that a qusifree state is defined by the formula

$$\varphi(W(f)) = \exp(-\tfrac{1}{2}\alpha(f,f)),$$

where the real bilinear form α satisfies the positivity condition (5.15). A qusifree state is analytic and derivation yields

$$\varphi(B(f)B(g)) = \alpha(f,g) - \mathrm{i}\sigma(f,g). \tag{5.20}$$

Proposition 5.19 *Let φ be a quasifree state on CCR(H,σ) given by (5.16) and $f_1, f_2, \dots, f_n \in H$. Then*

$$\varphi(B(f_n)B(f_{n-1})\dots B(f_1)) = 0$$

if n is odd. For an even n we have

$$\varphi(B(f_n)B(f_{n-1})\dots B(f_1)) = \sum \prod_{m=1}^{n/2} \left(\alpha(f_{k_m}, f_{j_m}) - i\sigma(f_{k_m}, f_{j_m})\right),$$

where the summation is over all partitions $\{H_1, H_2, \dots, H_{n/2}\}$ of $\{1, 2, \dots, n\}$ such that $H_m = \{j_m, k_m\}$ with $j_m < k_m$ $(m = 1, 2, \dots, n/2)$.

Proof. We benefit from the formula

$$\varphi(B(f_n)B(f_{n-1})\dots B(f_1)) = (-i)^n \frac{\partial^n}{\partial_n \dots \partial_1} \varphi(W(t_n f_n)\dots W(t_1 f_1)).$$

Since we have

$$\begin{aligned} &W(t_n f_n)W(t_{n-1}f_{n-1})\dots W(t_1 f_1) \\ &\qquad = W(f_n f_n + t_{n-1}f_{n-1} + \dots + t_1 f_1) \times \exp i \left(\sum_{l>k} t_l t_k \sigma(f_l, f_k) \right) \end{aligned}$$

(5.16) yields

$$\begin{aligned} \varphi(W(t_n f_n)\dots W(f_1 f_1)) &= \exp\left(-\frac{1}{2} \sum_{m=1}^{n} t_m^2 \alpha(f_m, f_m) \right) \\ &\exp\left(\sum_{l>k} t_l t_k(-\alpha(f_l, f_k) + i\sigma(f_l, f_k)) \right). \end{aligned} \tag{5.21}$$

What we need is the coefficient of $t_1 t_2 \dots t_n$ in the power series expansion. Such term comes only from the second factor of (5.21) and only in the case of an even n. In the claim it is described exactly the possibilities for getting $t_1 t_2 \dots t_n$ as a product of factors $t_l t_k$ $(l > k)$. □

By means of (5.20) we have also

$$\varphi(B(f_n)B(f_{n-1})\dots B(f_1)) = \sum \prod \varphi\left(B(f_{k_m})B(f_{j_m})\right), \tag{5.22}$$

were summation and product are similar to those in Proposition 5.19. The expression (5.22) makes clear that the value of a quasifree state φ on any polynomial of field operators is completely determined by the **two-point-functions** $\varphi(B(f)B(g))$ $(f, g \in H)$.

5.7 Purification

Let H be a real Hilbert space with inner product $(\,\cdot\,,\,\cdot\,)$ and let σ be a nondegenerate symplectic form on H such that

$$|\sigma(f,g)|^2 \le (f,f)(g,g) \qquad (f,g \in H) \tag{5.23}$$

holds. There exists a contraction D on H such that

$$\sigma(f,g) = (Df,g) \qquad (f,g \in H)\,. \tag{5.24}$$

Evidently $D^* = -D$. If $Df = 0$ then due to the nondegeneracy of σ $f = 0$ and hence D is invertible. Consider the polar decomposition

$$D = J|D|\,. \tag{5.25}$$

The property $D^* = -D$ gives that

$$J|D|J^* = -J^2|D|$$

and the uniqueness of the polar decomposition (applied for the positive operator $J|D|J^*$) guarantees that

$$-J^2 = I \quad \text{and} \quad J|D| = |D|J\,. \tag{5.26}$$

The state space of a C*-algebra is a compact convex subset of the dual space if it is endowed with the weak topology. A state is called **pure** if it is an extremal point of the state space.

Proposition 5.20 *Let φ be a quasifree state on $CCR(H,\sigma)$ so that*

$$\varphi(W(h)) = \exp\left(-\tfrac{1}{2}(h,h)\right) \qquad (h \in H),$$

If φ is pure then $|D|$ (given by (5.25) is the identity.

Proof. We shall argue by contradiction. Assume that there exists $f \in H$ such that

$$(|D|f,f) = 1 \quad \text{and} \quad (|D|^{-1} - I)^{1/2} f \ne 0\,. \tag{5.27}$$

Set $L = |D|^{1/2}(|D|^{-1} - I)^{1/2}$ and note that L is a contraction. We define a symmetric bilinear form as

$$S(g_1,g_2) = (g_1,g_2) - (Lg_1,|D|^{1/2}f)\cdot(Lg_2,|D|^{1/2}f)\frac{(Lf,|D|^{1/2}f)^2}{(Lf,Lf)}$$

and show that

$$S(g,g) \ge (|D|g,g) \qquad (g \in H) \tag{5.28}$$

or equivalently

$$((I - |D|)g,g)(Lf,Lf) \ge (Lg,|D|^{1/2}f)^2(Lf,|D|^{1/2}f)^2\,. \tag{5.29}$$

(5.29) is a consequence of the Schwarz inequality :

$$\begin{aligned}(Lf, |D|^{1/2}f)^2 &\leq (Lf, Lf)(|D|f, f) = (Lf, Lf) \\ (Lg, |D|^{1/2}f)^2 &\leq (L^2 g, g)(|D|f, f) = ((I - |D|)g, g) .\end{aligned}$$

By means of (5.24) and (5.25) we infer from (5.28) that

$$|\sigma(h, g)|^2 = (J|D|h, g)^2 \leq (|D|h, h)(|D|g, g) \leq S(h, h)S(g, g) .$$

Now Theorem 5.15 tells us that there is a (quasifree) state ω on $CCR(H, \sigma)$ such that

$$\omega(W(h)) = \exp\left(-\frac{1}{2}S(h, h)\right) . \qquad (h \in H)$$

We can see from Proposition 5.13 that if ω is any state of $CCR(H, \sigma)$ and F is a linear functional on H then there exists a state ω_F such that

$$\omega_F(W(h)) = \omega(W(h)) \exp(iF(h)) .$$

Writing a for $(Lf, |D|^{1/2}f)(Lf, Lf)^{-1/2}$ we set a state ω_λ for $\lambda \in \mathbb{R}$ as follows.

$$\omega_\lambda(W(h)) = \exp\left(-\frac{1}{2}S(h, h) + i\lambda(Lh, |D|^{1/2}f)a\right) \quad (h \in H) .$$

With the shorthand notation b for $(Lh, |D|^{1/2}f)$ we have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{-\lambda^2/2}}{(2\pi)^{1/2}} \omega_\lambda(W(h)) d\lambda &= e^{-\frac{1}{2}(h,h)} (2\pi)^{-1/2} \\ &\quad \times \int_{-\infty}^{\infty} \exp(-\lambda^2/2 + b^2a^2/2 + iba) d\lambda \\ &= e^{-\frac{1}{2}(h,h)} (2\pi)^{-1/2} \\ &\quad \times \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}(\lambda + iab)^2\right) d\lambda \\ &= e^{-\frac{1}{2}(h,h)}\end{aligned}$$

and this means that

$$\varphi = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp(-\lambda^2/2) \omega_\lambda d\lambda .$$

This decomposition is in contradiction with the starting assumption on φ. Hence the proof has been completed. □

Theorem 5.21 *Let the quasifree state φ defined on $CCR(H, \sigma)$ be given by a complete inner product $\alpha(\,\cdot\,,\,\cdot\,)$ as*

$$\varphi(W(h)) = \exp\left(-\tfrac{1}{2}\alpha(h, h)\right) .$$

Then φ is pure if and only if it is a Fock state.

This result makes Proposition 5.20 more complete. Remember that a state on a C*-algebra is pure if and only if the corresponding GNS-representation is irreducible (see [6], Thm. 2.3.19). Theorem 5.10 tells us that Fock states are pure and Proposition 5.20 yields that the other states are not so.

Now we are going to see that every quasifree state is a restriction of a Fock state of a bigger CCR-algebra.

Theorem 5.22 *Let $H_2 = H \oplus H$ be the direct sum Hilbert space and set a contraction D_2 of H_2 by the matrix*

$$D_2 = \begin{bmatrix} D & J\sqrt{I+D^2} \\ J\sqrt{I+D^2} & -D \end{bmatrix} . \tag{5.30}$$

Then the bilinear form

$$\sigma_2(W(f_2)) = \exp\left(-\frac{1}{2}\|f_2\|^2\right) \qquad (f_2, g_2 \in H_2)$$

is a symplectic form and the quasifree state

$$\varphi_2(W(f_2)) = \exp\left(-\frac{1}{2}\|f_2\|^2\right) \qquad (f_2 \in H_2) \tag{5.31}$$

on $CCR(H_2, \sigma_2)$ is a Fock state.

Proof. The proof is rather straightforward. We recall the relations

$$JD = DJ\ ,\ D^* = -D\ ,\ J = J^*\ ,\ J^2 = -id$$

These give that

$$D_2^* = -D_2 \quad \text{and} \quad D_2^2 = -id\,,$$

in other words, D_2 is a skewadjoint unitary. Hence σ_2 is an antisymmetric form and (4.9) defines a quasifree state. By the definition at the end of Chapter 3, φ_2 is a Fock state. □

Since

$$\sigma_2(f \oplus 0, f' \oplus 0) = (D_2(f \oplus 0), f' \oplus 0) = (Df, f') = \sigma(f, f')$$

the mapping

$$W(f) \mapsto W(f \oplus 0) \qquad (f \in H)$$

gives rise to an embedding of $CCR(H, \sigma)$ into $CCR(H_2, \sigma_2)$. Fock states are pure and that is the reason why the procedure described in Theorem 4.9 is called purification. Due to the direct sum $H_2 = H \oplus H$, doubling is another used term.

Purification is a standard way to reduce assertions on arbitrary quasifree states to those on Fock states. For example, we have

Corollary 5.23 *For an arbitrary quasifree state φ the linear manifold $\mathcal{D}_B^\varphi$ is dense in the GNS Hilbert space $\mathcal{H}_\varphi$ and consists of entire analytic vectors for every field operator $B_\varphi(f)$ $(f \in H)$.*

5.8 Exercises

1. Show that
$$\varphi(W(z)) = \exp\Big(-\frac{|z|^2}{2} - \langle z, Az\rangle\Big) \qquad (z \in \mathcal{H})$$
defines a state of $CCR(\mathcal{H})$ for a positive operator $A \in \mathcal{H}$.

2. Compute the density of the state
$$\varphi(W(z)) = \exp\Big(-\frac{|z|^2}{2} - \langle z, Az\rangle\Big)$$
when $\mathcal{H}$ has dimension 2 in the formalism of Example 5.12 and $A\eta_i = \lambda_i\eta_i$ $(i = 1, 2)$.

3. Show that the entropy of the previous state is
$$\mathrm{Tr}\,\kappa(A), \qquad \kappa(t) = -t\log t + (t+1)\log(t+1).$$

Chapter 6

Fluctuations and central limit

On matrix algebra $M_d(\mathbb{C})$ we fix a faithful state ψ, i.e. a state whose density matrix ρ_ψ is strictly positive definite. The algebra $M_d(\mathbb{C})$ becomes a complex Hilbert space with the inner product

$$\langle X, Y\rangle_\rho = \psi(Y^*X) \equiv \operatorname{Tr}(\rho_\psi Y^* X) \qquad (X, Y \in M_d(\mathbb{C})).$$

We consider the algebra $\mathrm{CCR}(M_d(\mathbb{C}))$ of the canonical commutation relation. On this algebra we have the Fock state φ defined by

$$\varphi(W(X)) = \exp\Big(-\tfrac{1}{2}\operatorname{Tr}\rho X^* X\Big).$$

By the GNS-construction, φ generates the Fock representation of the CCR-algebra.

The simplectic form is

$$\sigma(X, Y) = \frac{1}{2\mathrm{i}}\psi(X^*Y - Y^*X).$$

On the subspace $M_d(\mathbb{C})^{sa} \subset M_d(\mathbb{C})$ of self-adjoint matrices the simplectic form is

$$\sigma(X, Y) = \frac{1}{2\mathrm{i}}\psi([X, Y]) \qquad (X, Y \in M_d(\mathbb{C})^{sa}).$$

This simplectic form is degenerate on $M_d(\mathbb{C})^{sa}$. The real Hilbert space

$$L^2_{\mathbb{R}}(\rho) := (M_d(\mathbb{C}))^{sa}, \langle\,\cdot\,,\,\cdot\,\rangle_\rho)$$

is a direct sum of orthogonal subspaces: $H_\rho \oplus H_\rho^\perp$, where

$$H_\rho = \left\{A \in L^2_{\mathbb{R}}(\rho) : [A, \rho] = 0\right\}.$$

In particular, if $B = B_1 \oplus B_2 \in L^2_{\mathbb{R}}(\rho)$ then

$$\varphi(W(B)) = \exp\left(-\frac{1}{2}(B_1, B_1)_\rho\right)\exp\left(-\frac{1}{2}(B_2, B_2)_\rho\right). \tag{6.1}$$

Moreover, since $\sigma(A, B) = 0$ for $A \in \mathcal{H}_\rho$ and B arbitrary we get the following factorization

$$CCR(M(\mathbb{C}^d)^{sa}, \sigma) \cong CCR(\mathcal{H}_\rho, \sigma) \otimes CCR(\mathcal{H}_\rho^\perp, \sigma), \tag{6.2}$$

and by (6.1) the state φ factorizes as

$$\varphi = \varphi_1 \otimes \varphi_2. \tag{6.3}$$

The left side of the tensor product is a commutative algebra which is isomorphic to $L^\infty\left(\mathbb{R}^{|\mathcal{H}_\rho|}\right)$ carrying a Gaussian state with covariance $(A,B)_\rho$.

Consider the infinite tensorproduct C*-algebra $\mathcal{B} := \otimes_{i=1}^\infty \mathcal{B}_i$, where $\mathcal{B}_i$'s are copies of $M_d(\mathbb{C})$. Each $X \in M_d(\mathbb{C})$ will be indentified with

$$X \otimes I \otimes I \otimes \ldots$$

and so $M_d(\mathbb{C})$ may be considered to be a subalgebra of $\mathcal{B}$. The right shift endomorphism γ of $\mathcal{B}$ is determined by the property

$$\gamma : X_1 \otimes X_2 \otimes \ldots \otimes X_n \otimes I \otimes I \otimes \ldots \mapsto I \otimes X_1 \otimes \ldots \otimes X_n \otimes I \otimes \ldots .$$

On the language of algebraic probability $\mathcal{B}$ with the state $\omega = \psi \otimes \psi \otimes \ldots$ forms a probability space and $M_d(\mathbb{C}) \subset \mathcal{B}$ corresponds to a randon variable. Speaking this language

$$M_d(\mathbb{C}), \gamma(M_d(\mathbb{C})), \gamma^2(M_d(\mathbb{C})), \ldots$$

is a sequence of identically distributed independent randon variables, that is, a Bernoulli process (cf. [23]). For $X \in M_d(\mathbb{C})$

$$F_k(X) = \frac{1}{\sqrt{k}} \sum_{i=0}^{k-1} (\gamma^i(X) - \psi(X)) \tag{6.4}$$

is called the k^{th} fluctuation of X.

Theorem 6.1 *Let $A_1, \ldots, A_k \in M_d(\mathbb{C})^{sa}$ satisfying $\varphi(A_\ell) = 0$, for $1 \le \ell \le k$. Then we have the following*

$$\lim_{n\to\infty} \omega\left(\prod_{\ell=1}^k \exp(iF_n(A_\ell))\right) = \varphi\left(\prod_{\ell=1}^k W(A_\ell)\right).$$

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