7. Gyakorlat

(2) a) igen

b) nem: pl: $f(x)\equiv 1 \quad \forall x\in[0,1] \Rightarrow \langle f,f\rangle = \int_0^1 0^2\,dx = 0$ ✗

c) igen: $f,g\in C^1[0,1]$:

- $\langle f,g\rangle = \langle g,f\rangle$

  $\int_0^1 f'g' + f(0)g(0) \overset{?}{=} \int_0^1 g'f' + g(0)f(0)$ ✓

- $\langle f+g,h\rangle = \langle f,h\rangle + \langle g,h\rangle$

  $\int_0^1 (f+g)'h' + (f+g)(0)\cdot h(0) = \int_0^1 f'h' + g'h' + (f(0)+g(0))\cdot h(0)$

  $\overset{?}{=} \int_0^1 f'h' + f(0)h(0) + \int_0^1 g'h' + g(0)h(0)$ ✓

- $\lambda\langle f,g\rangle = \langle \lambda f,g\rangle$; $\lambda\in\mathbb{R}$ ← $\mathbb{R}$ feletti vektortér

  $\lambda\int_0^1 f'g' + \lambda\cdot f(0)\cdot g(0) \overset{?}{=} \int_0^1 \lambda f'g' + (\lambda f)(0)\cdot g(0)$ ✓

- $\int_0^1 f'^2\,dx + f^2(0) \geq 0 \Leftarrow f'(x)\geq 0$ ✓, $f^2(0)\geq 0$

- $\int_0^1 f'^2(x)\,dx + f^2(0) = 0 \Rightarrow f'(0)=0$ és $f(0)=0 \Rightarrow \underline{f\equiv 0}$ ✓

d) igen

e) nem: pl: $2\cdot\langle(1,0,0),(1,0,0)\rangle = 2\cdot 1 = 2$, $\langle 2(1,0,0),(1,0,0)\rangle = 4$ ✗

f) nem: pl: $\langle(0,1,0),(0,1,0)\rangle = 0$ ✗

g) igen

(4) b) Megoldás G-S-ort.-val

$\underline{w}=?$ $\quad \underline{w}\perp(\underline{v}_1,\underline{v}_2,\underline{v}_3) \Leftrightarrow \underline{w}\perp \mathrm{Span}(\underline{v}_1,\underline{v}_2,\underline{v}_3)$, $w_4 = 1$ ← 4. koord.-ja 1!

Keressünk Span $\underline{v}_1,\underline{v}_2,\underline{v}_3$-nak ONB-t!

$\underline{v}_1 \xrightarrow{G-S} \underline{w}_1 = \frac{\underline{v}_1}{|\underline{v}_1|} = \frac{1}{\sqrt5}\cdot(1,0,0,-2)$

$\underline{v}_2 \xrightarrow{G-S} \underline{w}_2 = \frac{\underline{v}_2 - \langle \underline{w}_1,\underline{v}_2\rangle\cdot\underline{w}_1}{|-\!\!-|} = \frac{(0,1,1,0) - 0\cdot\underline{w}_1}{\text{hossza}} = \frac{1}{\sqrt2}\cdot(0,1,1,0)$

$\underline{v}_3 \xrightarrow{G-S} \underline{w}_3 = \frac{v_3 - \langle w_1,v_3\rangle\cdot\underline{w}_1 - \langle \underline{w}_2,\underline{v}_3\rangle\cdot w_2}{|-\!\!-|} = \frac{\begin{pmatrix}1\\1\\-1\\0\end{pmatrix} - \frac15\cdot 1\cdot\begin{pmatrix}1\\0\\0\\-2\end{pmatrix} + \frac12\cdot 0\begin{pmatrix}0\\1\\1\\0\end{pmatrix}}{\text{sz. hossza}}$

$= \frac{(5,5,-5,0)-(1,0,0,-2)}{\text{számláló hossza}} = \frac{(4,5,-5,+2)}{\text{hossza}} = \frac{1}{\sqrt{70}}(4,5,-5,2)$

$\mathbb{R}^4$-ben 3 vektorra merőleges vektort csak 1 lin. ftlen van.

(Az összes így néz ki, hogy $c\cdot\underline{w}$ $(\underline{w}\neq 0)$

Válasszunk tetszőleges vektort $\mathbb{R}^4\setminus\mathrm{Span}(v_1,v_2,v_3)$-ból! Pl $\underline{v}_4=(1,0,0,0)$ vagy $\underline{v}_4=(1,1,1,1)$

$v_4 \xrightarrow{GS} w_4 = \frac{v_4 - \langle v_4|w_1\rangle\cdot\underline{w}_1 - \langle v_4,w_2\rangle\cdot\underline{w}_2 - \langle v_4|w_3\rangle w_3}{\text{hossza}} =$

$= (1,1,1,1) - \frac{1}{5}\cdot 1(1,0,0,-2) - \frac{1}{2}\cdot 2\cdot(0,1,1,0) - \frac{1}{70}\cdot 6\cdot(4,5,-5,2)$

$\frac{1}{7}(6,-3,3,3)$ hossz

$= \frac{(16, 2, 0, 50, 43)}{\text{hossz}} \longrightarrow$ egység, 4. koordinátájú

$\Rightarrow \underline{(2,-1,1,1)}$

2. megoldás: $\underline{w} = ?$ $\quad w \perp v_1,\ w \perp v_2,\ w \perp v_3,\ \langle w, (0,0,0,1)\rangle = 1$

Felírva az egyenletrendszert $\underline{w} = (w_1, w_2, w_3, w_4)$

$w \perp v_1 \iff v_1 \cdot w^T = 0 \ \dots$

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} w^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

kibővített mx:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \overset{S_3-S_1}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \overset{S_3-S_2}{\sim}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow w_4 = 1 \quad \overset{\text{3. egy}}{w_3 = 1} \quad \overset{\text{2. egy}}{w_2 = -1} \quad \overset{\text{1. egy}}{w_1 = 2}$$

$\Rightarrow \underline{\underline{(2,-1,1,1)}}$

d) Lehet G-S-sel ...

$w = ?$ $\quad |w| = 1 \leftarrow$ nem lineáris egyenlet

$\underline{w} \perp \underline{v}_1, \underline{v}_2, \underline{v}_3 \leftarrow$ lin. egyenletek: $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 3 \\ 1 & -1 & 1 & -1 \end{bmatrix}, \ \underline{w} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Homogén az egyenletrendszer, ezért az utolsó $\underline{0}$ oszlopot elhagyhatom.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 3 \\ 1 & -1 & 1 & -1 \end{bmatrix} \overset{S_2-2S_1,\ S_3-S_1}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & -2 & 0 & -2 \end{bmatrix} \overset{S_3-2S_2}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 2 & -4 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 2 & -4 & 0 \end{array}\right]$$

ált. mo. $w_4 = t$ $\quad$ 3. egy. $w_3 = 2t$ $\quad$ 2. egy. $w_2 = w_4 - w_3 = -t$ $\quad$ 1. egy. $w_1 = -w_2 - w_3 - w_4 = -2t$

$$\underline{w} = t \cdot \begin{bmatrix} -2 \\ -1 \\ 2 \\ 1 \end{bmatrix}$$

$\Rightarrow (-2t)^2 + (-1t)^2 + (2t)^2 + (1t)^2 = 1$ egyenletet kell megoldani

$10t^2 = 1 \Rightarrow t = \pm 1/\sqrt{10}$

$\pm\frac{1}{\sqrt{10}}\cdot(-2,-1,2,1)^T$ a 2 db mo.

f) Elég ortogonális bázis $\Rightarrow$ nem kell normálni!

$\underline{v}_1 = (1,2,1,3)$

$\underline{v}_2 = (4,1,1,1) \rightsquigarrow \underline{w}_2 = \underline{v}_2 - \frac{\langle v_1, v_2\rangle}{|v_1|^2} v_1 = (4,1,1,1) - \frac{10}{15}\cdot(1,2,1,3)$

$3\cdot\underline{w}_2 = (12,3,3,3) - 2(1,2,1,3) = \underline{(10,-1,1,3)} \leftarrow$ ez jó 2.-nak

$\underline{v}_3 = (3,1,1,0) \rightsquigarrow w_3 = \underline{v}_3 - \frac{\langle v_3, v_1\rangle \cdot v_1}{|v_1|^2} - \frac{\langle v_3, w_2\rangle \cdot w_2}{|w_2|^2} = (3,1,1,0) - \frac{6}{15}(1,2,1,3) - \frac{30}{111}(10,-1,1,3)$ $\quad \frac{10}{37}$

$37\cdot 5\cdot w_3 = (555,185,185,0) - (74,148,74,222) - (500,-50,50,150) =$

$185 \cdot \underline{w}_3 = (-19, 87, 61, -72)$

$w = (0, -1, 2, 1) \in \text{Span}(w_1, w_2, w_3)$

1. megoldás: $w \stackrel{?}{=} \frac{\langle w, w_1 \rangle}{|w_1|^2} w_1 + \frac{\langle w, w_2 \rangle}{|w_2|^2} w_2 + \frac{\langle w, w_3 \rangle}{|w_3|^2} w_3$

2. megoldás
Lin. egy. rsz-rel

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 2 & 1 & 1 & -1 \\ 3 & 1 & 1 & 2 \\ 3 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & -7 & -5 & -1 \\ 0 & -3 & -2 & 2 \\ 0 & -11 & -9 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & -1 & -1 & -5 \\ 0 & -3 & -2 & 2 \\ 0 & -11 & -9 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 17 \\ 0 & 0 & 2 & 56 \end{array}\right]$$ $\times$ nincs (*)

g) $f_1(x) = x^2 - 3, \quad f_2(x) = 2x - 5, \quad f_3(x) = x^3 - x$

$$f_1(x) = x^2 - 3 \xrightarrow{G-S} g_1(x) = \frac{x^2-3}{\sqrt{\int_0^1 (x^2-3)^2 dx}} = \frac{x^2-3}{\sqrt{\left[\frac{x^5}{5} - 2x^3 + 9x\right]_0^1}} = \frac{x^2-3}{\sqrt{7 + 1/5}} = \frac{\sqrt{5}}{6}(x^2-3)$$

$$f_2(x) = 2x - 5 \xrightarrow{G-S} g_2(x) = \underbrace{2x - 5 - \frac{5}{36} \cdot \int_0^1 (2x-5)(x^2-3)\,dx \cdot (x^2-3)}_{\text{„hossza"}}$$

$$\int_0^1 2x^3 - 5x^2 - 6x + 15 = \left[\frac{x^4}{2} - \frac{5}{3}x^3 - 3x^2 + 15x\right]_0^1 = 12{,}5 - 5/3 = \frac{65}{6}$$

$$g_2(x) = -\frac{105 - 432x + 325x^2}{6\sqrt{43}}$$

$$f_3(x) = x^3 - x \rightsquigarrow g_3(x) = \underbrace{x^3 - x - \frac{5}{36} \int_0^1 (x^2-3) \cdot (x^3-x)\,dx\,(x^2-3) - \ldots}_{\text{hossza}}$$

$$\underline{- \frac{1}{36 \cdot 43} \cdot \int_0^1 -(105 - 432x + 325x^2) \cdot (x^3 - x)\,dx \cdot (-105 + 432x - 325x^2)}$$

$$= \underbrace{(x^3 - x) - \frac{5}{36} \cdot \frac{2}{3}(x^2-3) + \frac{1}{36 \cdot (43)} \cdot \frac{1657}{12} \cdot (-105 + 432x - 325x^2)}_{\text{hossza}}$$

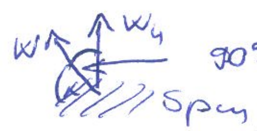
$$= \underbrace{-\frac{55415}{6708} + \frac{19325x}{559} - \frac{541105}{201124}x^2 + x^3}_{\text{hossza}} = \ldots$$

(*) Mivel nincs benne $w \notin \text{Span}(w_1, v_2, v_3)$

Meg kell határoznunk a szöget

w_4 legyen olyan, hogy $w_4 \perp \text{Span}\, v_1, v_2, v_3$ → egy. mo. / → Gram-Schmidt order

$w_4 = (-1, -5, 8, 1)$



$$\cos(90° - \alpha) = \frac{\langle w_4, w \rangle}{|w_4| \, |w|} = \frac{12}{\sqrt{6} \cdot \sqrt{91}}$$

Vagy: $\text{Span}(w_1, w_2, w_3)$-ban w-hez legközelebbi vektor „ $\underline{u} = \sum_{i=1}^{3} \langle w, w_i \rangle \cdot w_i$

$$\cos \alpha = \frac{\langle w, u \rangle}{|w| \cdot |u|}$$