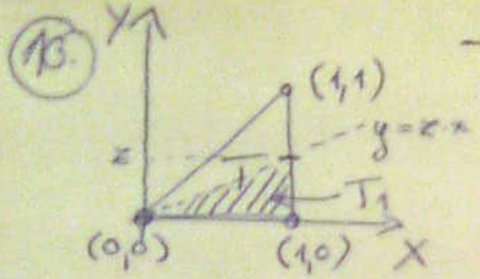
13.



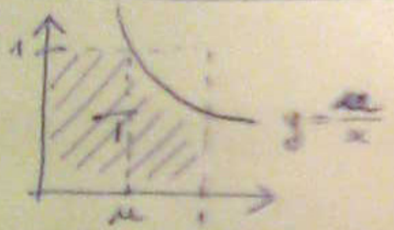
T-n egyenletes $\rightsquigarrow$ $f(x,y) = \begin{cases} \frac{1}{\text{terület}} = 2 & (x,y) \in T \\ 0 & \text{egyébként} \end{cases}$

$Z = \frac{Y}{X} \in [0,1]$, ha $(X,Y) \in T$

$\mathbb{P}\left(\frac{Y}{X} < z\right) = \mathbb{P}(Y < z \cdot X) = T_1 \text{ területe} \cdot 2 = \frac{z}{2} \cdot 2 = z \rightsquigarrow Z$ egyenletes $[0,1]$-n

egyenletesség miatt $\qquad f(x,y)$

11.



$f(x,y) = \begin{cases} 1 & (x,y) \in [0,1]^2 \\ 0 & \text{egyébként} \end{cases}$

$U = X \cdot Y \in [0,1]$

$\mathbb{P}(X \cdot Y < u) = \mathbb{P}\left(Y < \frac{u}{X}\right) = T \text{ területe} = \int_0^u 1\,dx + \int_u^1 \frac{u}{x}\,dx =$

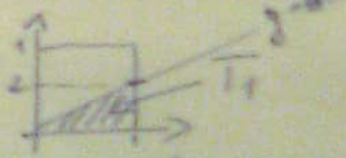
egyenletesség

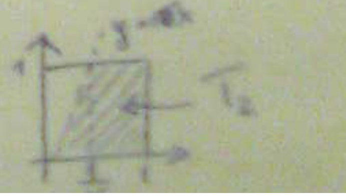
$= u - u \ln u = H(u) \quad u \in (0,1)$, 0 egyébként

$\rightsquigarrow h(u) = H'(u) = \ln u \quad u \in (0,1)$, 0 egyébként

$\mathbb{P}\left(\frac{Y}{X} < v\right) = \mathbb{P}(Y < v \cdot X) = \begin{cases} \iint_{T_1} f(x,y)\,dx\,dy = \frac{v}{2}, & v \in [0,1] \\ \iint_{T_2} f(x,y)\,dx\,dy = 1 - \frac{1}{2v}, & v \in (1,\infty) \end{cases}$

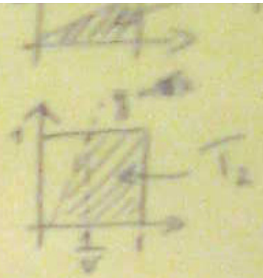
$L(v)$, $V \in [0,\infty)$





$$\mathbb{P}\left(\frac{Y}{X} < v\right) = \mathbb{P}(Y < v\cdot X) = \begin{cases} \iint_{T_1} f(x,y)\,dx\,dy = \frac{v}{2}, & v\in[0,1] \\ \iint_{T_2} f(x,y)\,dx\,dy = 1-\frac{1}{2v}, & v\in(1,\infty) \end{cases}$$

$L(v)$, $V\in[0,\infty)$



$$\ell(v) = L'(v) = \begin{cases} \frac{1}{2}, & v\in[0,1] \\ \frac{1}{2v^2}, & v>1 \end{cases}$$

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(12) $f(x,y) = 4xy$, $(x,y)\in[0,1]^2$, 0 egyébként

$\rightsquigarrow U = X\cdot Y \in [0,1]$ Megj.: a 11. feladat ábráit és jelöléseit használjuk

$V = Y/X \in [0,\infty)$

$$H(u) = \mathbb{P}(XY<u) = \mathbb{P}\left(Y < \frac{u}{x}\right) = \iint_{T_1} f(x,y)\,dx\,dy = \int_0^u\int_0^1 4xy\,dy\,dx + \int_u^1\int_0^{u/x} 4xy\,dy\,dx =$$

$$= 4\cdot\int_0^u x\,dx\cdot\int_0^1 y\,dy + \int_u^1 4x\int_0^{u/x} y\,dy\,dx = 4\cdot\frac{u^2}{2}\cdot\frac{1}{2} + \int_u^1 4x\cdot\frac{u^2}{2x^2}\,dx = u^2 - u^2\ln u = u^2(1-\ln(u))$$

ha $u\in(0,1]$ és 0 egyébként

$h(u) = H'(u) = -4u\ln(u)$ ha $u\in(0,1]$ és 0 egyébként.

$$L(v) = \mathbb{P}(Y/X < v) = \mathbb{P}(Y < v\cdot X) \Rightarrow$$

ha $v\in[0,1] \rightsquigarrow L(v) = \iint_{T_1} f(x,y)\,dx\,dy = \int_0^1\int_0^{v\cdot x} 4xy\,dy\,dx = \int_0^1 4x\cdot\left[\frac{y^2}{2}\right]_0^{v\cdot x} dx =$

$$= 2\int_0^1 v^2x^3\,dx = \frac{2}{4}\cdot v^2 = \frac{v^2}{2}$$

⑫ Folyt.

ha $v \in (1,\infty) \rightsquigarrow L(v) = \iint_{T_v} f(x,y)\,dx\,dy = \int_0^{1/v}\int_0^{v\cdot x} 4xy\,dy\,dx + \int_{1/v}^{1}\int_0^{1} 4xy\,dy\,dx =$

$$= \int_0^{1/v} 4x\cdot\left[\frac{y^2}{2}\right]_0^{v\cdot x} dx + \int_{1/v}^{1} 4x\cdot\left[\frac{y^2}{2}\right]_0^{1} dx = 2v^2\int_0^{1/v} x^3\,dx + 2\int_{1/v}^{1} x\,dx = \cancel{2}\,v^2\,\frac{1}{4_2}\,\frac{1}{v^4} + 2\left(\frac{1}{2} - \frac{1}{2v^2}\right) =$$

$$= 1 - \frac{1}{2v^2}$$

$$\rightsquigarrow f(v) = L'(v) = \begin{cases} v, & v \in [0,1] \\ \frac{1}{v^3}, & v > 1 \end{cases}$$

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⑬ $X \sim Exp(\lambda)$, $Y \sim Exp(\mu)$ $\rightsquigarrow f(x,y) \underset{\text{függetlenség}}{=} \lambda e^{-\lambda x}\mu e^{-\mu y}$ $\quad x \geq 0,\ y \geq 0$

$V = Y/X \in [0,\infty)$

$$f(v) = \int_0^\infty f(x, vx)\,x\,dx = \int_0^\infty \lambda\cdot\mu\cdot e^{-\lambda x - \mu v x}\,x\,dx = \lambda\mu\int_0^\infty x\,e^{-(\lambda+\mu v)x}\,dx =$$

$$= \lambda\cdot\mu\underbrace{\left[\frac{x\,e^{-(\lambda+\mu v)x}}{-(\lambda+\mu v)}\right]_0^\infty}_{0} + \frac{\lambda\mu}{\lambda+\mu v}\int_0^\infty e^{-(\lambda+\mu v)x}\,dx = \frac{\lambda\mu}{(\lambda+\mu v)^2}\cdot\underbrace{\left[-e^{-(\lambda+\mu v)x}\right]_0^\infty}_{1} =$$

$$= \frac{\lambda\mu}{(\lambda+\mu v)^2}$$ ha $v \in [0,\infty)$, 0 egyébként.

$$\mathbb{P}(\underset{\geq 0}{X} < \underset{\geq 0}{Y}) = \mathbb{P}\left(1 < \frac{Y}{X}\right) = \int_1^\infty f(v)\,dv = \int_1^\infty \frac{\lambda\mu}{(\lambda+\mu v)^2}\,dv = \lambda\cdot\mu\left[-\frac{1}{\lambda+\mu v}\right]_1^\infty = \frac{\lambda\mu}{\lambda+\mu}$$