1. Bizonyítsuk be, hogy a következő sorozatok monoton nőnek.
   a) $a_0 = 0,\ a_{n+1} = a_n^2 + 1$ b) $b_0 = 4,\ b_{n+1} = \frac{b_n^2+6}{5}$ c) $b_0 = 1,\ b_{n+1} = \frac{b_n^2+6}{5}$

2. Határozzuk meg a határértéket és a küszöbindexet a megadott ε-hoz:
   a) $\lim_{n\to\infty} \frac{1}{2n},\ \varepsilon = 10^{-6}$ b) $\lim_{n\to\infty} \frac{n+3}{2n+3},\ \varepsilon = 0,01$ c) $\lim_{n\to\infty} \frac{n^2+6n}{3n^2+5},\ \varepsilon = 0,001$
   d) $\lim_{n\to\infty} \frac{n+\sqrt{n}}{2n-4},\ \varepsilon = 0,01$ e) $\lim_{n\to\infty} \frac{3^n}{3^{n+1}+2},\ \varepsilon = 10^{-4}$ f) $\lim_{n\to\infty} \frac{n^3+5}{2n^3},\ \varepsilon = 10^{-5}$

2. a) $\lim_{n\to\infty} \frac{1}{2n} = 0$. Mely n-ekre teljesül, hogy

$$\varepsilon = 10^{-6} > |\frac{1}{2n} - 0| = \frac{1}{2n}.$$

Ez pedig ekvivalens azzal, hogy $2n > \frac{1}{10^{-6}} = 10^6$. Tehát $N = 500000$ lesz a küszöbindex.

2. b) $\lim_{n\to\infty} \frac{n+3}{2n+3} = \lim_{n\to\infty} \frac{1+\frac{3}{n}}{2+\frac{3}{n}} = \frac{1}{2}$. Vizsgáljuk, hogy mikor teljesül: $\varepsilon = 0,01 > |\frac{n+3}{2n+3} - \frac{1}{2}| = |\frac{n+3-(n+1,5)}{2n+3}| = |\frac{1,5}{2n+3}| = \frac{1,5}{2n+3}$. Ez pedig ekvivalens azzal, hogy $2n+3 > \frac{1,5}{0,01} = 150$. Tehát $N = 73,5$ lesz a küszöbindex.

2. c) $\lim_{n\to\infty} \frac{n^2+6n}{3n^2+5} = \lim_{n\to\infty} \frac{1+\frac{6}{n}}{3+\frac{5}{n^2}} = \frac{1}{3}$. Vizsgáljuk, hogy mikor teljesül: $\varepsilon = 0,001 > |\frac{n^2+6n}{3n^2+5} - \frac{1}{3}| = |\frac{n^2+6n-(n^2+5/3)}{3n^2+5}| = |\frac{6n-5/3}{3n^2+5}| = \frac{6n-5/3}{3n^2+5}$. (Abszolút érték elhagyható, ha $n > 2$.) Ez pedig ekvivalens azzal, hogy $3n^2 + 5 > \frac{6n-5/3}{0,001} = 6000n - 5000/3$, azaz $0 < n^2 - 2000n + 5015/3 = (n-1000)^2 - 1000^2 + 5015/3 \Leftarrow 0 < (n-1000)^2 - 1000^2$. (Számológép használatával minimális küszöbindexet is megkaphattunk, vagy másodfokú megoldóképletet is alkalmazhattuk volna.) Teljesül, ha ($n < 0$ és) $n > 2000$. A küszöbindexnek jó az $N = 2000$.

2. d) $\lim_{n\to\infty} \frac{n+\sqrt{n}}{2n+4} = \lim_{n\to\infty} \frac{1+\frac{1}{\sqrt{n}}}{2+\frac{4}{n}} = \frac{1}{2}$. Vizsgáljuk, hogy mikor teljesül: $\varepsilon = 0,01 > |\frac{n+\sqrt{n}}{2n+4} - \frac{1}{2}| = |\frac{n+\sqrt{n}-(n+2)}{2n+4}| = |\frac{\sqrt{n}-2}{2n+4}| = \frac{\sqrt{n}-2}{2n+4}$. (Abszolút érték elhagyható, ha $n > 1$.) Ez pedig ekvivalens azzal, hogy $2n + 4 > \frac{\sqrt{n}+2}{0,01} = 100\sqrt{n} + 200$. Azaz $n - 50\sqrt{n} - 96 > 0$, ($\sqrt{n}_{1,2} = \frac{\pm\sqrt{50^2+4\cdot 96}+50}{2} = \pm\sqrt{25^2+96} + 25$), azaz $n > 51,85^2 \sim 2688$ Tehát $N = 2688$ lesz a küszöbindex.

2. e) $\lim_{n\to\infty} \frac{3^n}{3^{n+1}+2} = \lim_{n\to\infty} \frac{1}{3+\frac{2}{3^n}} = \frac{1}{3}$. Vizsgáljuk, hogy mikor teljesül: $\varepsilon = 10^{-4} > |\frac{3^n}{3^{n+1}+2} - \frac{1}{3}| = |\frac{3^n-3^n-2/3}{3^{n+1}+2}| = |\frac{-2/3}{3^{n+1}+2}| = \frac{2/3}{3^{n+1}+2}$. Ez pedig ekvivalens azzal, hogy $3^{n+1} + 2 > \frac{2/3}{10^{-4}} = 20000/3$. Azaz $n > \log_3(19994/3) - 1 \sim 8 - 1$. Tehát $N = 7$ lesz a küszöbindex.

2. f) $\lim_{n\to\infty} \frac{n^3+5}{2n^3} = \lim_{n\to\infty} \frac{1+\frac{5}{n^3}}{2} = \frac{1}{2}$. Vizsgáljuk, hogy mikor teljesül: $\varepsilon = 10^{-5} > |\frac{n^3+5}{2n^3} - \frac{1}{2}| = |\frac{n^3+5-n^3}{2n^3}| = |\frac{5}{2n^3}| = \frac{5}{2n^3}$. Ez pedig ekvivalens azzal, hogy $2n^3 > \frac{5}{10^{-5}} = 500000$. Azaz $n > \sqrt[3]{500000} \sim 79$. Tehát $N = 79$ lesz a küszöbindex.

3. Állapítsuk meg, hogy a sorozat konvergens-e vagy divergens. Ha konvergens határozzuk meg a határértékét.

♣ $a_n = \frac{1-n}{1+2n}$ ♣ $b_n = \frac{n^2-3n}{n^2+6n-5}$ ♣ $c_n = \frac{1+2+\cdots+n}{n^2}$ ♣ $d_n = \frac{n^3+n^2+1}{n^2+1}$

♣ $e_n = \frac{1-5n^4}{n^4+8n^3}$ ♣ $f_n = \frac{n-1}{n^2-2n+1}$ ♣ $g_n = \frac{(-1)^{n+1}}{2n^2-1}$ ♣ $h_n = \sqrt{\frac{2n}{n+1}}$

♣ $i_n = n^3/\binom{n}{2}$ ♣ $j_n = \frac{n!}{2^n 3^n}$ ♣ $k_n = \frac{2^n+3^n}{n+4^n}$ ♣ $l_n = \frac{5^n+6^n}{6^{n+1}+4^n}$

♢ $m_n = (1+\frac{1}{n})^n$ ♢ $n_n = (\frac{n}{n+1})^n$ ♢ $o_n = (1-\frac{1}{n^2})^n$ ♢ $p_n = (1+\frac{1}{2n})^{n+5}$

♢ $q_n = (1+\frac{4\sqrt{n}}{n+9})^{5\sqrt{n}+2}$ ♢ $r_n = (\frac{3n+12}{3n+7})^{6n} + 2$ ♢ $s_n = (1+\frac{3}{2^n+5})^{2^n-1}$ ♢ $t_n = (\frac{n+1}{n+2})^{n^2}$

♡ $u_n = \sqrt[n]{n^4+3n^2+1}$ ♡ $v_n = \sqrt[n]{10n}$ ♡ $w_n = 2\sqrt[n]{4^{n-1}n+4}$ ♡ $x_n = \sqrt[n]{n} + \sqrt[n]{3}$

♡ $y_n = \sqrt[n]{\sin(3^n}$ ♡ $z_n = \sqrt[n]{\frac{2^n}{3^n+23n}} + \frac{2}{3}$ ♡ $aa_n = \sqrt[n]{2}$ ♡ $ab_n = \sqrt[n+1]{n+5}$

♠ $ac_n = n - \sqrt{n^2-n}$ ♠ $ad_n = \sqrt{n^2+n} - \sqrt{n^2-1}$ ♠ $ae_n = \sqrt[3]{n^2+n} - \sqrt[3]{n^2-n}$

4. Az azonos jellel jelöltek:

$A = \lim_{n\to\infty} \frac{1-n}{1+2n} = \lim_{n\to\infty} \frac{\frac{1}{n}-1}{\frac{1}{n}+2} = \frac{1}{2}$ $B = \lim_{n\to\infty} \frac{n^2-3n}{n^2+6n-5} = \lim_{n\to\infty} \frac{1-\frac{3}{n}}{1+\frac{6}{n}-\frac{5}{n^2}} = 1$

$C = \lim_{n\to\infty} \frac{1+2+\cdots+n}{n^2} = \lim_{n\to\infty} \frac{\frac{(n+1)n}{2}}{n^2} = \lim_{n\to\infty} \frac{n^2+n}{2n^2} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2} = \frac{1}{2}$

$D = \lim_{n\to\infty} \frac{n^3+n^2+1}{n^2+1} = \lim_{n\to\infty} \frac{n+1+\frac{1}{n^2}}{1+\frac{1}{n^2}} = \infty$ divergens

$E = \lim_{n\to\infty} \frac{1-5n^4}{n^4+8n^3} = \lim_{n\to\infty} \frac{\frac{1}{n^4}-5}{1+\frac{8}{n}} = -\frac{5}{8}$ $F = \lim_{n\to\infty} \frac{n-1}{n^2-2n+1} = \lim_{n\to\infty} \frac{\frac{1}{n}-\frac{1}{n^2}}{1-\frac{2}{n}+\frac{1}{n^2}} = 0$

$G = \lim_{n\to\infty} \frac{(-1)^{n+1}}{2n^2-1} = 0$ $H = \lim_{n\to\infty} \sqrt{\frac{2n}{n+1}} = \lim_{n\to\infty} \sqrt{\frac{2}{1+\frac{1}{n}}} = \sqrt{2}$

$I = \lim_{n\to\infty} n^3 / \left(\begin{array}{c} n \\ 2 \end{array} \right) = \lim_{n\to\infty} \frac{n^3}{\frac{n(n-1)}{2}} = \lim_{n\to\infty} \frac{2n^3}{n^2-n} = \lim_{n\to\infty} \frac{2n}{1-\frac{1}{n}} = \infty$ divergens

$J = \lim_{n\to\infty} \frac{n!}{2^n 3^n} \geq \lim_{n\to\infty} \frac{1\cdot\ldots1\cdot\frac{n}{2}\ldots\frac{n}{2}}{6^n} = \lim_{n\to\infty} \frac{1^{\frac{n}{2}}\cdot(\frac{n}{2})^{\frac{n}{2}}}{6^n} = \lim_{n\to\infty}(\frac{\frac{n}{2}}{36})^{\frac{n}{2}} = \lim_{n\to\infty}(\frac{n}{72})^{\frac{n}{2}} = \infty$

$0 \leq K = \lim_{n\to\infty} \frac{2^n+3^n}{n+4^n} \leq \lim_{n\to\infty} \frac{3^n+3^n}{4^n} = \lim_{n\to\infty} 2(\frac{3}{4})^n = 0$ rendőrelv miat $K = 0$.

$L = \lim_{n\to\infty} \frac{(\frac{5}{6})^n+1}{6+(\frac{4}{6})^n} = \frac{1}{6}$ $\qquad M = \lim_{n\to\infty}(1+\frac{1}{n})^n = e$

$N = \lim_{n\to\infty}(\frac{n}{n+1})^n = \lim_{n\to\infty}(\frac{(n+1)-1}{n+1})^n = \lim_{n\to\infty}(1-\frac{1}{n+1})^n = \lim_{n\to\infty}[(1-\frac{1}{n+1})^{n+1}]^{\frac{n}{n+1}} = e^1$

$O = \lim_{n\to\infty}(1-\frac{1}{n^2})^n = \lim_{n\to\infty}[(1-\frac{1}{n^2})^{n^2}]^{\frac{1}{n}} = e^0 = 1$

$P = \lim_{n\to\infty}(1+\frac{1}{2n})^{n+5} = \lim_{n\to\infty}[(1+\frac{1}{2n})^{2n}]^{\frac{n+5}{2n}} = e^{\frac{1}{2}} = \sqrt{e}$

$Q = \lim_{n\to\infty}(1+\frac{4\sqrt{n}}{n+9})^{5\sqrt{n}+2} = \lim_{n\to\infty}[(1+\frac{4\sqrt{n}}{n+9})^{\frac{n+9}{4\sqrt{n}}}]^{\frac{4\sqrt{n}}{n+9}\cdot(5\sqrt{n}+2)} = \lim_{n\to\infty} e^{\frac{20n+8\sqrt{n}}{n+9}} = e^{20}$

$R = \lim_{n\to\infty}(\frac{3n+12}{3n+7})^{6n} + 2 = \lim_{n\to\infty}[(1+\frac{5}{3n+7})^{\frac{3n+7}{5}}]^{\frac{30n}{3n+7}} + 2 = e^{\frac{30n}{3n+7}} + 2 = e^{10} + 2$

$S = \lim_{n\to\infty}(1+\frac{3}{2^n+5})^{2^n-1} = \lim_{n\to\infty}[(1+\frac{3}{2^n+5})^{\frac{2^n+5}{3}}]^{\frac{3}{2^n+5}(2^n-1)} = \lim_{n\to\infty} e^{\frac{3-\frac{1}{2^n}}{1+\frac{5}{2^n}}} = e^3$

$T = \lim_{n\to\infty}(\frac{n+1}{n+2})^{n^2} = \lim_{n\to\infty}[(1-\frac{1}{n+2})^{-n^2}]^{-1} = e^{-1}$

$U = \lim_{n\to\infty} \sqrt[n]{n^4+3n^2+1} > \lim_{n\to\infty} \sqrt[n]{n^4} = \lim_{n\to\infty} \sqrt[n]{n}^4 = 1^4 = 1$

$V = \lim_{n\to\infty} \sqrt[n]{10n} = \lim_{n\to\infty} \sqrt[n]{10}\cdot \sqrt[n]{n} = 1\cdot 1 = 1$

$W = \lim_{n\to\infty} 2\sqrt[n]{4^{n-1}n+4} \geq \lim_{n\to\infty} 2\sqrt[n]{4^{n-1}n} = \lim_{n\to\infty} 2\sqrt[n]{4^{n-1}}\cdot \sqrt[n]{n} = \lim_{n\to\infty} 2\sqrt[n]{4^n}\cdot \sqrt[n]{\frac{1}{4}}\cdot 1 =$ $2\cdot 4 = 8$ és $W \leq \lim_{n\to\infty} 2\sqrt[n]{4^{n-1}n+4^{n-1}n} = \lim_{n\to\infty} 2\sqrt[n]{2\cdot 4^{n-1}n} = \lim_{n\to\infty} 2\sqrt[n]{\cdot 4^n}\sqrt[n]{\frac{1}{2}}\sqrt[n]{n} = 8$, azaz a rendőrelv miatt $(8 \leq W \leq 8)$ $W = 8$.

$X = \lim_{n\to\infty} \sqrt[n]{n} + \sqrt[n]{3} = 1+1 = 2$

$Y = \lim_{n\to\infty} \sqrt[n]{\sin(3^n)}$ $Z = \lim_{n\to\infty} \sqrt[n]{\frac{2^n}{3^n+23n}} + \frac{2}{3} = \lim_{n\to\infty} \sqrt[n]{\frac{(\frac{2}{3})^n}{1+\frac{23n}{3^n}}} + \frac{2}{3} = \lim_{n\to\infty} \sqrt[n]{(\frac{2}{3})^n} + \frac{2}{3} = \frac{4}{3}$

$AA = \lim_{n\to\infty} \sqrt[n]{2} = 1$

$AB = \lim_{n\to\infty} \sqrt[n+1]{n+5} \leq \lim_{n\to\infty} \sqrt[n+1]{n+1+4\cdot(n+1)} = \lim_{n\to\infty} \sqrt[n+1]{5}\sqrt[n+1]{n+1} = 1\cdot 1 = 1$

$AC = \lim_{n\to\infty} n-\sqrt{n^2-n} = \lim_{n\to\infty} n-\sqrt{n^2-n}\frac{n+\sqrt{n^2-n}}{n+\sqrt{n^2-n}} = \lim_{n\to\infty} \frac{n^2-(n^2-n)}{n+\sqrt{n^2-n}} = \lim_{n\to\infty} \frac{1}{1+\sqrt{1-\frac{1}{n}}} =$ $\frac{1}{1+1} = \frac{1}{2}$

$AD = \lim_{n\to\infty} \sqrt{n^2+n}-\sqrt{n^2-1} = \lim_{n\to\infty} \sqrt{n^2+n}-\sqrt{n^2-1}\frac{\sqrt{n^2+n}+\sqrt{n^2-1}}{\sqrt{n^2+n}+\sqrt{n^2-1}} = \lim_{n\to\infty} \frac{n^2+n-n^2+1}{\sqrt{n^2+n}+\sqrt{n^2-1}} =$ $\lim_{n\to\infty} \frac{1+\frac{1}{n}}{\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n^2}}} = \frac{1}{1+1} = \frac{1}{2}$

$AE = \lim_{n\to\infty} \sqrt[3]{n^2+n} - \sqrt[3]{n^2-n} = \lim_{n\to\infty} \sqrt[3]{n^2+n} - \sqrt[3]{n^2-n}\frac{\sqrt[3]{n^2+n}^2+\sqrt[3]{n^2-n}\sqrt[3]{n^2+n}+\sqrt[3]{n^2-n}^2}{\sqrt[3]{n^2+n}^2+\sqrt[3]{n^2-n}\sqrt[3]{n^2+n}+\sqrt[3]{n^2-n}^2} =$ $\lim_{n\to\infty} \frac{n^2+n-n^2+n}{\sqrt[3]{n^2+n}^2+\sqrt[3]{n^2-n}\sqrt[3]{n^2+n}+\sqrt[3]{n^2-n}^2} = 0$ Vizsgáljuk meg a legnagyobb tarokat felül: $2n$, alul: $\max(\sqrt[3]{n^2}^2, \sqrt[3]{n^2}\sqrt[3]{n^2}, \sqrt[3]{n^2}^2) = \sqrt[3]{n^2}^2 = n^{\frac{4}{3}} > n$.

4. Döntsük el a következő sorozatokra, hogy korlátos, monoton illetve konvergens-e. Ha konvergens, akkor adjuk meg a határértékét.
   a) $a_n = \frac{3n+1}{n+1}$ $\quad$ b) $b_n = \frac{2^n 3^n}{n!}$ $\quad$ c) $c_n = \frac{n^2+5n}{n^2+1}$ $\quad$ d) $d_n = \frac{2^n+1}{n!}$

5. Határozzuk meg a torlódási pontjait a következő sorozatoknak:
   a) $\cos(n\frac{\pi}{2})\frac{2n^2-3}{n^2+n+8}$ $\quad$ b) $(-1)^n(1+\frac{1}{n})$ $\quad$ c) $\sqrt{\frac{n^3+(-1)^n n^3}{3n^3+n+8}}$ $\quad$ d) $(-1)^n\cos(n\pi)$