I. $a_n = \frac{2n^2}{n^2+1}$ | $\varepsilon = 0{,}001$

$N_0 = !$

$$\lim_{n\to\infty}\left(\frac{2n^2}{n^2+1}\right) = \lim_{n\to\infty}\left(\frac{2}{1+\frac{1}{n^2}}\right) = \underline{2}$$

$\frac{1}{n^2} \to 0$

$|a_n - 2| < 0{,}001$

<u>Sejtés</u>: a_n mon növő: $a_{n+1} \geq a_n \ \forall n$

$$\frac{2n^2+4n+2}{n^2+2n+2} = \frac{2(n+1)^2}{(n+1)^2+1} \geq \frac{2n^2}{n^2+1}$$

$$(n^2+1)(2n^2+4n+2) \geq 2n^2(n^2+2n+2)$$

$$\underline{2n^4} + \underline{4n^3} + \underline{4n^2} + 4n + 2 \geq \underline{4n^2} + \underline{4n^3} + \underline{4n^2}$$

$$4n+2 \geq 0$$

$$n \geq -\frac{1}{2} \checkmark$$

$\Rightarrow$ MON. NÖVŐ!

$|a_n - 2| < 0{,}001$

Mivel a_n mon. nő $\Rightarrow a_n \le 2 \Rightarrow a_n - 2 \le 0$

$|a_n - 2| < 0{,}001$

$\Downarrow (a_n - 2) \le 0$

$2 - a_n < 0{,}001$

$1{,}999 < \frac{2n^2}{n^2+1}$

$1{,}999n^2 + 1{,}999 < 2n^2$

$\cancel{0{,}001n^2 - 1{,}999}$

$1{,}999 < 0{,}001n^2$

$1999 < n^2$

$\Rightarrow n_{1,2} = \pm\sqrt{1999} \approx \pm 44{,}71$

$\Rightarrow \boxed{N_0 = 45}$

Ezzel pontosan $n=45$-től lesz igaz! De bármely $N_0 \ge 45$ szám jó megoldás!

Ellenőrzésképpen, ha megnézitek: $a_{44} \approx 1{,}9989$, $a_{45} \approx 1{,}9990$

II. $a_n = \left(\frac{4n+2}{4n-3}\right)^{3-2n} = \left(\frac{4n-3+5}{4n-3}\right)^{3-2n} = \left(1+\frac{5}{4n-3}\right)^{3-2n}$

$\lim\limits_{n\to\infty} a_n = e^c$, ahol $c = \lim\limits_{n\to\infty} \frac{5\cdot(3-2n)}{4n-3} = \lim\limits_{n\to\infty}\left(\frac{15-10n}{4n-3}\right) = \lim\limits_{n\to\infty}\left(\frac{\underbrace{\frac{15}{n}}_{\to 0}-10}{4-\underbrace{\frac{3}{n}}_{\to 0}}\right) = \underline{\underline{-\frac{5}{2}}}$

$\Rightarrow \lim\limits_{n\to\infty} a_n = e^{-\frac{5}{2}}$

$b_n = \left(\frac{4^n}{4^n+2^n+1}\right)^{2^{n+1}+n} = \left(\frac{4^n+2^n+1-2^n-1}{4^n+2^n+1}\right)^{2^{n+1}+n} =$

$= \left(1+\frac{-2^n-1}{4^n+2^n+1}\right)^{2^{n+1}+n}$

$\lim\limits_{n\to\infty} b_n = e^c$, ahol $c = \lim\limits_{n\to\infty}\left(\frac{(-2^n-1)\cdot(2^{n+1}+n)}{4^n+2^n+1}\right) =$

$= \lim\limits_{n\to\infty}\left(\frac{-2\cdot 4^n - 2^{n+1} - n2^n - n}{4^n+2^n+1}\right) = \lim\limits_{n\to\infty}\left(\frac{-2-\underbrace{2\cdot\left(\frac{1}{2}\right)^n}_{\to 0}-\underbrace{n\cdot\left(\frac{1}{2}\right)^n}_{\to 0}-\underbrace{\frac{n}{4^n}}_{\to 0}}{1+\underbrace{\left(\frac{1}{2}\right)^n}_{\to 0}+\underbrace{\frac{1}{4^n}}_{\to 0}}\right) =$

$= -2 \Rightarrow \lim\limits_{n\to\infty} b_n = \underline{\frac{1}{e^2}}$

$$c_n = \sqrt{n^2+5n+1} - n = \left(\sqrt{n^2+5n+1} - n\right)\left(\frac{\sqrt{n^2+5n+1}+n}{\sqrt{n^2+5n+1}+n}\right) =$$

$$= \frac{n^2+5n+1-n^2}{\sqrt{n^2+5n+1}+n} = \frac{5n+1}{\sqrt{n^2+5n+1}+n} = \frac{5+\frac{1}{n}}{\sqrt{1+\frac{5}{n}+\frac{1}{n^2}}+1} \xrightarrow[n \to \infty]{} \underline{\underline{\frac{5}{2}}}$$

$$d_n = \left(\frac{3n+\sqrt{n}}{3n+1}\right)^{n^2} = \left(\frac{3n+1+\sqrt{n}-1}{3n+1}\right)^{n^2} = \left(1+\frac{\sqrt{n}-1}{3n+1}\right)^{n^2} \xrightarrow[n \to \infty]{} e^c,$$

ahol $c = \lim\limits_{n\to\infty}\left(\frac{\sqrt{n}\cdot(n^2-1)}{3n+1}\right) = \lim\limits_{n\to\infty}\left(\frac{n^{\frac{5}{2}}-\sqrt{n}}{3n+1}\right) = \infty$, azaz

$$\underline{\underline{\lim_{n\to\infty} d_n = \infty}}$$

III. 1/

$$\left(\left(\ln(\ln(x))\right)^{e^{\cos x}}\right)' = \left(e^{\overbrace{e^{\cos x}}^{g} \cdot \overbrace{\ln(\ln(\ln(x)))}^{f}}\right)' =$$

$$= \ln(\ln(x))^{e^{\cos x}} \cdot \left(\underbrace{e^{\cos x}\cdot -\sin x}_{g'} \cdot \underbrace{\ln(\ln(\ln(x)))}_{f} + \underbrace{e^{\cos x}}_{g} \cdot \underbrace{\frac{1}{\ln(\ln(x))}\cdot\frac{1}{\ln(x)}\cdot\frac{1}{x}}_{f'}\right)$$

2/ $$\left(\left((\sin x)^{\cos x}\right)^{\operatorname{arctg} x}\right)' = \left((\sin x)^{\cos x \cdot \operatorname{arctg} x}\right)' = \left(e^{\overbrace{\cos x \cdot \operatorname{arctg} x}^{f} \cdot \overbrace{\ln(\sin x)}^{g}}\right)' =$$

$$= (\sin x)^{\cos x \cdot \operatorname{arctg} x} \cdot \left(\underbrace{\left(-\sin x \cdot \operatorname{arctg} x + \cos x \cdot \frac{1}{1+x^2}\right)}_{f'} \cdot \underbrace{\ln(\sin x)}_{g} + \underbrace{\cos x \cdot \operatorname{arctg} x}_{f} \cdot \underbrace{\frac{1}{\sin x} \cdot \cos x}_{g'} \right)$$

3*/ $$\left((\sin x)^{\left((\cos x)^{\operatorname{arctg} x}\right)}\right)' = \left(e^{\overbrace{(\cos x)^{\operatorname{arctg} x}}^{f} \cdot \overbrace{\ln(\sin x)}^{g}}\right)' =$$

$$(\sin x)^{\left((\cos x)^{\operatorname{arctg} x}\right)} \cdot \left(\underbrace{\cos x^{\operatorname{arctg} x} \cdot \left(\frac{\ln(\cos x)}{1+x^2} + \frac{\operatorname{arctg} x \cdot -\sin x}{\cos x} \right)}_{f'} \cdot \underbrace{\ln(\sin x)}_{g} + \underbrace{\cos x^{\operatorname{arctg} x}}_{f} \cdot \frac{1}{\underbrace{\sin x}_{g}} \cdot \cos x \right)$$

$$f' = \left(e^{\operatorname{arctg} x \cdot \ln(\cos x)}\right)' = \cos x^{\operatorname{arctg} x} \cdot \left(\frac{1}{1+x^2} \cdot \ln(\cos x) + \operatorname{arctg} x \cdot \frac{1}{\cos x} \cdot -\sin x \right)$$