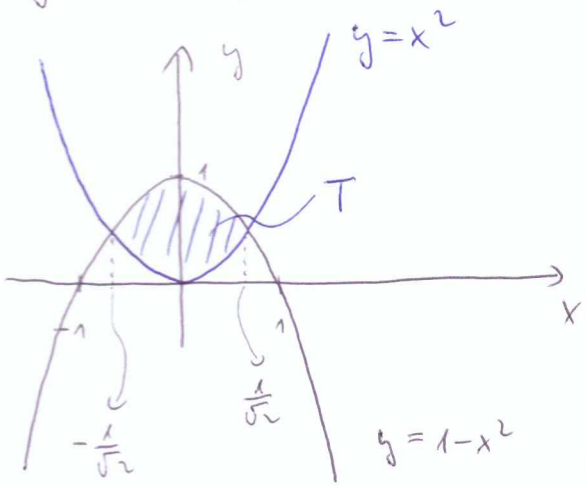
1/ (3) Területszámítás

a) $\left.\begin{array}{l} y = x^2 \\ y = 1 - x^2 \end{array}\right\}$ által bezárt korlátos terület



metszéspontok kiszámítása:

$$x^2 = 1 - x^2$$

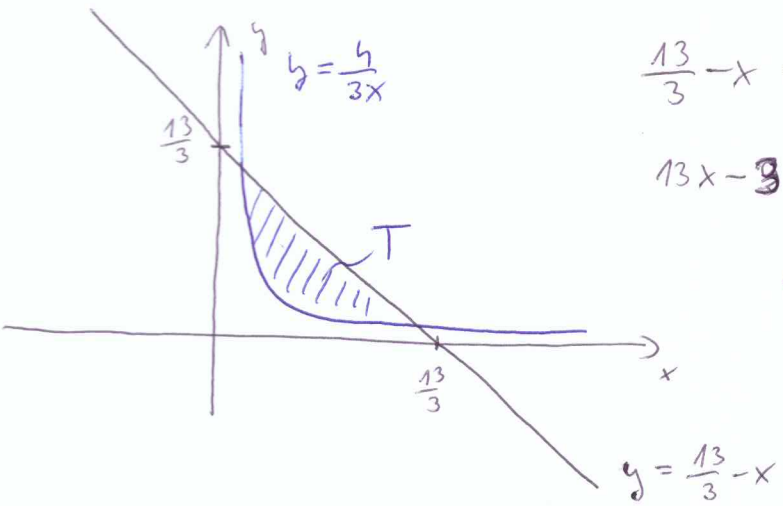
$$\hookrightarrow \quad x^2 = \frac{1}{2} \quad \rightsquigarrow \quad x_{1/2} = \pm \frac{1}{\sqrt{2}}$$

$$T = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1 - x^2 - x^2)\,dx = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1 - 2x^2)\,dx \underset{\uparrow}{=} 2 \cdot \int_0^{1/\sqrt{2}} (1 - 2x^2)\,dx =$$

a tartomány szimmetrikus az y tengelyre

$$= 2\left[x - 2\frac{x^3}{3}\right]_0^{1/\sqrt{2}} = 2\left(\frac{1}{\sqrt{2}} - 2\,\frac{\frac{1}{2\sqrt{2}}}{3}\right) = 2\left(\frac{1}{\sqrt{2}} - \frac{1}{3\sqrt{2}}\right) = \underline{\underline{\frac{4}{3\sqrt{2}}}}$$

b) $\left.\begin{array}{l} y = \frac{4}{3x} \\ y = \frac{13}{3} - x \end{array}\right\}$ köré zárt korlátos terület



metszéspontok kiszámítása:

$$\frac{13}{3} - x = \frac{4}{3x}$$

$$13x - 3x^2 = 4$$

$$9x^2 - 13x + 4 = 0$$

$$D = 169 - 4 \cdot 4 \cdot 9 = 25$$

$$x_{1/2} = \frac{13 \pm 5}{18} = \begin{cases} 1 \\ \frac{4}{9} \end{cases}$$

2)

$$3x^2 - 13x + 4 = 0$$

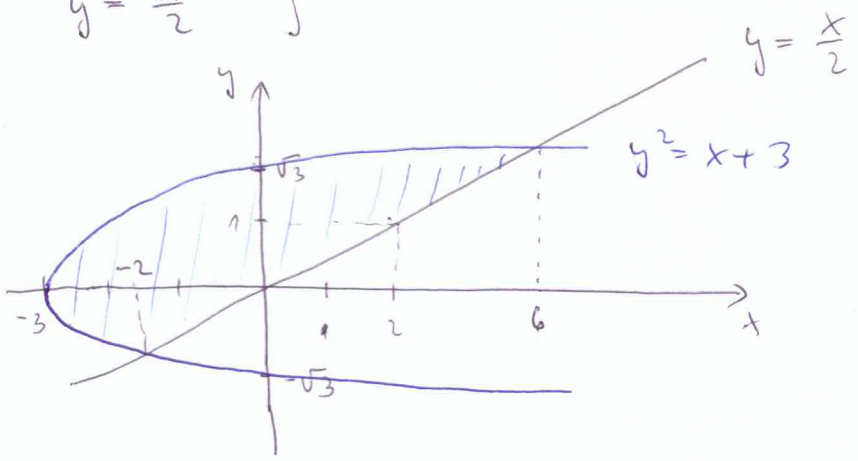
diszkrimináns: $D = 169 - 4\cdot 3\cdot 4 = 121$

$$x_{1,2} = \frac{13 \pm 11}{6} = \begin{cases} 4 \\ \frac{1}{3} \end{cases}$$

$$T = \int_{1/3}^{4} \left(\frac{13}{3} - x - \frac{4}{3x} \right) dx = \left[\frac{13}{3}x - \frac{x^2}{2} - \frac{4}{3}\ln|x| \right]_{1/3}^{4} =$$

$$= \frac{13}{3}\cdot 4 - \frac{16}{2} - \frac{4}{3}\ln 4 - \frac{13}{3}\cdot\frac{1}{3} + \frac{1}{18} + \frac{4}{3}\ln\frac{1}{3} = \ldots\ldots$$

c) $\left.\begin{array}{l} y^2 = x+3 \\ y = \frac{x}{2} \end{array}\right\}$ kiírt korlátos terület



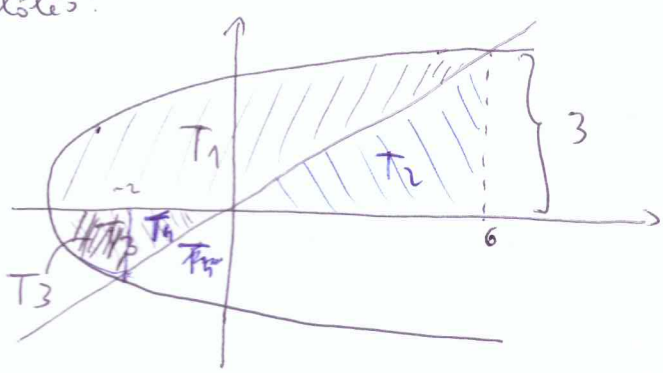
metszéspontok: $y = \frac{x}{2} \rightsquigarrow y^2 = \frac{x^2}{4} \Rightarrow x + 3 = \frac{x^2}{4}$

$\hookrightarrow x^2 - 4x - 12 = 0$

$(x-6)(x+2) = 0$

$$\boxed{\begin{array}{l} x_1 = -2 \\ x_2 = 6 \end{array}}$$

Jelölés:



$x_2 = 6 \rightsquigarrow y_2 = \frac{x_2}{2} = \frac{6}{2} = 3$

$x_1 = -2 \rightsquigarrow y_1 = \frac{x_1}{2} = -1$

$T_2 = \frac{6\cdot 3}{2} = 9$ (Δ terület)

3)

$$T_1 = \int_{-3}^{6} \sqrt{x+3}\,dx - T_2 = \left[\frac{(x+3)^{3/2}}{3/2}\right]_{-3}^{6} - 9 =$$

$$= \frac{2}{3}\left(9^{3/2} - 0\right) - 9 = \frac{2}{3}\cdot 27 - 9 = \underline{\underline{9}}$$

$$T_3 = -\int_{-3}^{-2} -\sqrt{x+3}\,dx = \int_{-3}^{-2} \sqrt{x+3}\,dx = \left[\frac{(x+3)^{3/2}}{3/2}\right]_{-3}^{-2} =$$

előjeles terület

$$= \frac{2}{3}\left(1^{3/2} - 0\right) = \underline{\underline{\frac{2}{3}}}$$

$$T_4 = \frac{1\cdot 2}{2} = 1$$

Δ terület

$$\Rightarrow \quad T = T_1 + T_3 + T_4 = 9 + \frac{2}{3} + 1 = \underline{\underline{10 + \frac{2}{3}}}$$

d)

$$\left.\begin{aligned} \sqrt{x} + \sqrt{y} &= 1 \\ x + y &= 1 \end{aligned}\right\} \text{ háromszög alakú terület}$$

$\sqrt{x} + \sqrt{y} = 1 \rightsquigarrow \sqrt{y} = 1 - \sqrt{x}$ $\qquad x \geq 0, y \geq 0$

$\hookrightarrow \quad x \leq 1$ (hiányban ellentmondás)

$\Rightarrow \quad y = (1-\sqrt{x})^2 \quad , (0 \leq x \leq 1)$

másik egyenletből: $y = 1 - x$

$\Rightarrow$ metszéspontok: $(1-\sqrt{x})^2 = 1 - x = (1-\sqrt{x})(1+\sqrt{x})$

4)

$\hookrightarrow \quad (1-\sqrt{x})\left[1+\sqrt{x}-(1-\sqrt{x})\right]=0$

$(1-\sqrt{x})\cdot 2\sqrt{x}=0 \iff x=0 \text{ v. } x=1$

Mivel $[0,1]$-n $\quad 1-x \geq (1-\sqrt{x})^2$

$$\hookrightarrow \quad T=\int_0^1 \underbrace{\left[(1-x)-(1-\sqrt{x})^2\right]}_{1-x-1+2\sqrt{x}-x\,=\,-2x+2\sqrt{x}} dx = 2\int_0^1 (\sqrt{x}-x)\,dx \ominus$$

$$\ominus \; 2\left[\frac{x^{3/2}}{3/2}-\frac{x^2}{2}\right]_0^1 = 2\left(\frac{2}{3}-\frac{1}{2}\right)=\frac{4}{3}-1=\underline{\underline{\frac{1}{3}}}$$

e)

$$\left.\begin{array}{l} y=\ln x \\ y=\ln^2 x \end{array}\right\} \text{ közti korlátos terület}$$

metszéspontok: $\ln x=\ln^2 x \rightsquigarrow \ln x(1-\ln x)=0$ $\nearrow \ln x=0 \iff x_1=1$, $\searrow \ln x=1 \iff x_2=e$

mivel $[1,e]$-n $0\leq \ln x\leq 1 \Rightarrow$

$\ln x \geq \ln^2 x$, ha $x\in[1,e]$

$$\Rightarrow \quad T=\int_1^e (\ln x-\ln^2 x)\,dx = \underbrace{\int_1^e \ln x\,dx}_{①} - \underbrace{\int_1^e \ln^2 x\,dx}_{②}$$

külön-külön kiszámoljuk a tagokat:

① <u>parciális integrálással:</u>

$$\int_1^e \ln x\,dx = \int_1^e 1\cdot\ln x\,dx \overset{P}{=} \left[x\ln x\right]_1^e - \int_1^e x\cdot\frac{1}{x}\,dx \ominus$$

$u'=1 \rightsquigarrow u=x$

$v=\ln x \quad v'=\frac{1}{x}$

5)

$$\ominus \left[x \ln x\right]_1^e - \left[x\right]_1^e = e \underset{1}{\ln e} - 1 \cdot \underset{0}{\ln 1} - (e-1) = e-(e-1) = \underline{\underline{1}}$$

② helyettesítéses integrálással:

$$\int_1^e \ln^2 x \, dx \underset{\uparrow}{=} \int_0^1 t^2 e^t \, dt \quad \ominus$$

$$t := \ln x \rightsquigarrow x = e^t \rightsquigarrow \frac{dx}{dt} = e^t \rightsquigarrow \boxed{dx = e^t \, dt}$$

mivel az integrálban:

$$1 \le x \le e \overset{\ln}{\rightsquigarrow} \ln 1 \le \ln x \le \ln e$$

$$0 \le t \le 1$$

$\Uparrow$

az helyettesítésben megváltoznak az új változó határai

$$\ominus \left[t^2 e^t\right]_0^1 - \int_0^1 2t e^t \, dt =$$

$\uparrow$ parc. int.

$u = t^2 \rightsquigarrow u' = 2t$

$v' = e^t \rightsquigarrow v = e^t$

$\uparrow$ parc. int

$u = 2t \rightsquigarrow u' = 2$

$v' = e^t \rightsquigarrow v = e^t$

$$= \left[t^2 e^t\right]_0^1 - \left\{ \left[2t e^t\right]_0^1 - \underbrace{\int_0^1 2 e^t \, dt}_{\left[2e^t\right]_0^1} \right\} =$$

$$= \left[t^2 e^t\right]_0^1 - \left[2t e^t\right]_0^1 + \left[2e^t\right]_0^1 = e - 0 - 2e + 0 + 2e - 2 = \underline{\underline{e-2}}$$

$$\Rightarrow \quad T = 1 - (e-2) = \underline{\underline{3-e}}$$