1/

Határozott integrál

Def: $[a,b]$ intervallum felosztása olyan $\overset{F}{\overset{\shortparallel}{\{x_0, x_1, \dots, x_n\}}}$ véges pontsorozat, melyre:

$$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b.$$

F felosztás finomsága: $\max\limits_{i \in \{1,2,\dots,n\}} (x_i - x_{i-1})$

Def. $f : [a,b] \to \mathbb{R}$ korlátos, $F = \{x_0, \dots, x_n\}$ $[a,b]$ felosztása

$$\left.\begin{aligned} m &:= \inf_{x\in[a,b]} f(x), \quad m_i := \inf_{x \in [x_{i-1}, x_i]} f(x) \\ M &= \sup_{x \in [a,b]} f(x), \quad M_i := \sup_{x \in [x_{i-1}, x_i]} f(x) \end{aligned}\right\}$$

jól definiáltak + végesek

$s_F := \sum\limits_{i=1}^{n} m_i (x_i - x_{i-1})$ Darboux-féle alsó int. közelítő összeg

$S_F := \sum\limits_{i=1}^{n} M_i (x_i - x_{i-1})$ –"– felső –"–

$\sigma_F := \sum\limits_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$, $\xi_i \in [x_{i-1}, x_i]$ tetsz.

Riemann-féle ~~k~~ int. közelítő összeg

TÉTEL $\forall$ F felosztásra

$$m(b-a) \le s_F \le \sigma_F \le S_F \le M(b-a)$$

Biz. $m \le m_i \le f(\xi_i) \le M_i \le M$

$$\hookrightarrow m(b-a) = \sum_{i=1}^{n} m(x_i - x_{i-1}) \le s_F = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \le \sigma_F = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

$$\le \sum_{i=1}^{n} M_i (x_i - x_{i-1}) = S_F \le \sum_{i=1}^{n} M(x_i - x_{i-1}) = M(b-a) \quad \circ \, !$$

Def. F_1, F_2 $[a,b]$ felosztásai, F_1 $\forall$ osztópontja F_2-nek is osztópontja

$\Rightarrow$ F_2 az F_1 finomítása.

TÉTEL: Ha F_2 az F_1 finomítása $\Rightarrow$ $s_{F_1} \le s_{F_2}$ és $S_{F_2} \le S_{F_1}$.

Biz. $F_1 := \{x_0, \dots, x_n\}$ vegyük hozzá x^* új osztópontot:

$$x_{k-1} < x^* < x_k$$

$$\hookrightarrow s = m_1(x_1 - x_0) + \dots + m_{k-1}(x_{k-1} - x_{k-2}) + m_{k,1}(x^* - x_{k-1}) + m_{k,2}(x_k - x^*) + \dots$$

$$\dots + m_{k+1}(x_{k+1} - x_k) + \dots + m_n(x_n - x_{n-1}), \text{ ahol}$$

$$m_{k,1} = \inf_{x \in [x_{k-1}, x^*]} f(x), \quad m_{k,2} = \inf_{x \in [x^*, x_k]} f(x)$$

$$m_k \le m_{k,1} \text{ és } m_k \le m_{k,2}$$

$$\hookrightarrow m_k(x_k - x_{k-1}) = m_k(x_k - x^*) + m_k(x^* - x_{k-1}) \le m_{k,2}(x_k - x^*) + m_{k,1}(x^* - x_{k-1})$$

$\hookrightarrow$ $s_{F_1} \le s$ $\rightsquigarrow$ s_{F_2} megkapható véges lépésekben

ugyanígy S-re $\circ$!

2/

TÉTEL $[a,b]$ $\forall$ F_1, F_2 felosztására:

$$s_{F_1} \le S_{F_2}$$

Biz. F_3 : F_1 s F_2 ~~...~~ egyesítése $\rightsquigarrow$ F_3 F_1 s F_2-nek $\rightsquigarrow$ finomítása

$$\hookrightarrow \; s_{F_1} \le s_{F_3} \le S_{F_3} \le S_{F_2} \quad \circ !$$

f korlátos $\Rightarrow$ $\{ s_F : F \; [a,b] \text{ felosztása} \}$ korlátosak
$\{ S_F : \quad - \| - \quad \}$

$$\Downarrow$$

Def. $f : [a,b] \to \mathbb{R}$ korlátos fv

$\underline{I} := \sup \{ s_F \mid F \; [a,b] \text{ felosztása} \}$ Darboux-féle alsóintegrál

$\overline{I} := \inf \{ S_F : F \; [a,b] \text{ felosztása} \}$ Darboux-féle felső int

$$\Downarrow$$

$$m(b-a) \le \underline{I} \le \overline{I} \le M(b-a)$$

Def. $f : [a,b] \to \mathbb{R}$ korlátos fv Riemann-integrálható $[a,b]$-n, ha

$\underline{I} = \overline{I}$. Ekkor $\underline{I} = \overline{I} =: \int_a^b f(x)\,dx = \int_a^b f$

Ha $f : [a,b] \to \mathbb{R}$ nem korlátos $[a,b]$-n, vagy korlátos, de $\underline{I} < \overline{I}$

$\hookrightarrow$ f nem R-integrálható $[a,b]$-n

Megj: geom. jelentés: előjeles terület

Pl) Dirichlet-fv.
$$f(x)=\begin{cases}1, \text{ ha } x\in\mathbb{Q}\\ 0, \text{ ha } x\notin\mathbb{Q}\end{cases}$$

f korlátos, $\forall F$ felosztásra $s_F=0$, $S_F=1$

$\leadsto$ $\underline{I}=0$ és $\overline{I}=1$ $\Rightarrow$ f nem integrálható $[0,1]$-en.

Def: f integrálható $[a,b]$-n (jel: $f\in\mathcal{R}[a,b]$)

$$\int_b^a f(x)\,dx := -\int_a^b f(x)\,dx$$

TÉTEL: $f,g\in\mathcal{R}[a,b]$, $\lambda\in\mathbb{R}$

(i) cf, $f+g$, $f-g\in\mathcal{R}[a,b]$

$$\int_a^b \lambda f=\lambda\int_a^b f,\quad \int_a^b (f\pm g)=\int_a^b f\pm\int_a^b g;$$

ii) $[\alpha,\beta]\subset[a,b]$ $\Rightarrow$ $f\in\mathcal{R}[\alpha,\beta]$

iii) $a<c\leq b$ $\leadsto$ $\int_a^b f=\int_a^c f+\int_c^b f$

iv) $f(x)\leq g(x)\ \forall\ x\in[a,b]$ $\Rightarrow$ $\int_a^b f(x)\,dx\leq\int_a^b g(x)\,dx$

v) $|f|\in\mathcal{R}[a,b]$

$$\left|\int_a^b f(x)\,dx\right|\leq\int_a^b |f(x)|\,dx$$

vi) $\int_a^a f(x)\,dx=0$

3/ TÉTEL (Newton–Leibniz)

$f : [a,b] \to \mathbb{R}$ korlátos, integrálható $[a,b]$-n

$F : [a,b] \to \mathbb{R}$ f primitív f-e $[a,b]$-n

$\hookrightarrow \quad \int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$

Biz. $\quad B := \{a = x_0, x_1, \dots, x_n = b\}$ beosztás

$\forall k \in \{1, 2, \dots, n\} \rightsquigarrow$ Lagrange-féle középértéktétel F-re és $[x_{k-1}, x_k]$-n:

$$\exists\, \xi_k \in (x_{k-1}, x_k) : \frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}} = F'(\xi_k) = f(\xi_k)$$

$\hookrightarrow \quad F(x_k) - F(x_{k-1}) = f(\xi_k)(x_k - x_{k-1})$

$\hookrightarrow \quad F(b) - F(a) \underset{\text{teleszkóp}}{=} [F(x_1) - F(x_0)] + [F(x_2) - F(x_1)] + \dots + [F(x_n) - F(x_{n-1})] =$

$= f(\xi_1)(x_1 - x_0) + f(\xi_2)(x_2 - x_1) + \dots + f(\xi_n)(x_n - x_{n-1}) =$

$= \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) = \sigma_B$

de $\quad s_B \le \sigma_B = F(b) - F(a) \le S_B$

$\Downarrow$

$\underline{I} \le F(b) - F(a) \le \overline{I}$

f integrálható $\Rightarrow \quad \int_a^b f(x)\,dx = F(b) - F(a)$

Megj: $[F(x)]_a^b = [F(x)+c]_a^b \quad \forall c$ konstans.

Integrálható függvények

TÉTEL $f: [a,b] \to \mathbb{R}$ korlátos. Ha $\forall \varepsilon > 0$-hoz $\exists B$ beosztás, hogy $\underbrace{S_B - s_B}_{\text{oszcillációs összeg}} < \varepsilon$, akkor f integrálható $[a,b]$-n.

Biz: Ha $\forall \varepsilon > 0$-hoz $\exists B$ beosztás, hogy $S_B - s_B < \varepsilon$

$$\hookrightarrow \quad \overline{I} \le S_B < s_B + \varepsilon \le \underline{I} + \varepsilon$$

$$\Downarrow$$

$$0 \le \overline{I} - \underline{I} < \varepsilon \quad \forall \varepsilon > 0\text{-ra}$$

$$\Downarrow$$

$$\underline{I} = \overline{I}$$

4/

TÉTEL $f: [a,b] \to \mathbb{R}$ monoton $[a,b]$-n $\Rightarrow f \in \mathcal{R}[a,b]$

Biz tfh $f \nearrow$

Zárt intervallumon monoton nő' fv korlátos $\rightsquigarrow f(a) \le f(x) \le f(b) \quad \forall x \in [a,b]$

$\forall [\alpha,\beta] \subset [a,b] \rightsquigarrow f(\alpha) := \min_{x \in [\alpha,\beta]} f(x)$, $f(\beta) := \max_{x\in[\alpha,\beta]} f(x)$

Osszuk fel $[a,b]$-t n egyenlő részre:

$$\hookrightarrow s_n = f(x_0)(x_1-x_0) + f(x_1)(x_2-x_1) + \dots + f(x_{n-1})(x_n - x_{n-1}) =$$

$$= [f(x_0) + \dots + f(x_{n-1})] \frac{b-a}{n}$$

$$S_n = f(x_1)(x_1-x_0) + f(x_2)(x_2-x_1) + \dots + f(x_n)(x_n-x_{n-1}) =$$

$$= [f(x_1) + f(x_2) + \dots + f(x_n)] \frac{b-a}{n}$$

$\forall \varepsilon > 0 \rightsquigarrow$

$$S_n - s_n = [f(x_n) - f(x_0)] \frac{b-a}{n} = \frac{(f(b)-f(a))(b-a)}{n} < \varepsilon$$

elég nagy n-re

$\Downarrow$ T.

f integrálható.

f monoton csökkenésre ugyanígy

o!

TÉTEL $f:[a,b]\to\mathbb{R}$ folytonos $[a,b]$-n $\Rightarrow$ f integrálható $[a,b]$-n

Biz. f folytonos $\Rightarrow$ f korlátos $[a,b]$-n

+ f egyenletesen folytonos $[a,b]$-n

$$\Downarrow$$

$\forall\, \varepsilon>0$ pl. $\frac{\varepsilon}{b-a}$-hoz is $\exists\, \delta>0$, hogy $\forall\, x,y\in[a,b]$,

$|x-y|<\delta \Rightarrow |f(x)-f(y)|<\frac{\varepsilon}{b-a}$

Legyen B δ-nál finomabb felosztás: $B=\{x_0,x_1,\dots,x_n\}$

$$\max_{i\in\{1,\dots,n\}} (x_i-x_{i-1})<\delta$$

f folytonos $\rightsquigarrow$ $[x_{i-1},x_i]$-n felveszi minimumát vegyük ξ_i helyen

maximumát vegyük η_i helyen

$$\Downarrow$$

$$m_i=f(\xi_i)\quad,\ M_i=f(\eta_i)$$

$|\xi_i-\eta_i|<\delta$

$$\hookrightarrow\ S_B-s_B=\sum_{i=1}^{n}(M_i-m_i)(x_i-x_{i-1})=\sum_{i=1}^{n}\left[f(\eta_i)-f(\xi_i)\right](x_i-x_{i-1})=$$

$$=\sum_{i=1}^{n}|f(\eta_i)-f(\xi_i)|\,(x_i-x_{i-1})<\sum_{i=1}^{n}\frac{\varepsilon}{b-a}(x_i-x_{i-1})=$$

$$=\frac{\varepsilon}{b-a}\sum_{i=1}^{n}(x_i-x_{i-1})=\frac{\varepsilon}{b-a}(b-a)=\varepsilon. \rightsquigarrow f \text{ int.ható} \ \circ\,!$$

5/

<u>Def</u>: $f \in R[a,b] \rightsquigarrow \frac{1}{b-a}\int_a^b f(x)\,dx$ f integrálközepe

<u>TÉTEL</u> (integrálszámítás középértéktétele)

$$f \in C[a,b] \Rightarrow \exists\, \xi \in [a,b],\ \text{hogy}$$

$$f(\xi) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

<u>Biz</u>: $m := \min_{x\in[a,b]} f(x)$, $M := \max_{x\in[a,b]} f(x)$

$$\hookrightarrow m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

$$\hookrightarrow m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M$$

f folytonos $\Rightarrow \exists\, \xi \in [a,b]$, hogy $f(\xi) = \frac{1}{b-a}\int_a^b f(x)\,dx$

Közbülső felvesz m és M között

<u>Def</u>: $f \in R[c,d]$, $a \in [c,d]$

$$\hookrightarrow F(x) := \int_a^x f(t)\,dt \quad x \in [c,d] \quad f \text{ integrálfüggvénye}$$

TÉTEL $f \in C(I)$, $a \in I$, $F(x) = \int_a^x f(t)\,dt$, $x \in I$.

$\hookrightarrow$ F diff.ható I-n és $F'(x) = f(x)$ $\forall x \in I$

Biz. $x, x+h \in I$

$h \neq 0 \Rightarrow$

$$f \text{ folytonos} \Rightarrow \frac{F(x+h)-F(x)}{h} = \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right)$$

$$= \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(\xi) \quad \text{valamely } \xi \in [x, x+h]$$

$$\downarrow h \to 0 \quad \Rightarrow \xi \to x$$

$$\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} f(\xi(h)) = f(x) = F'(x)$$

o.k.

Köv $\forall f$ folytonos függvénynek $\exists$ primitív f-e

Pl
$$\lim_{x\to 0}\frac{\int_0^x \cos t^2\,dt}{x}, \quad \lim_{x\to 0}\frac{\int_0^x \ln(1+t)\,dt}{x^2}$$

Pl
$$f(x) = \int_0^x \frac{t^2+t-2}{t^2+1}\,dt$$ mely $\exists$-e $[0,\sqrt{3}]$-beli lok. szé-e?

Pl
$$F(x) = \int_{x^2}^{x^3} \frac{dt}{\sqrt{1+t^5}} \qquad F'(x) = ?$$

P1)

$$F(x) = \int_0^x \sin t^2 \, dt \quad \rightsquigarrow F'(x) = \sin x^2$$

$$G(x) = \int_0^{x^3} \sin t^2 \, dt = F(x^3) \rightsquigarrow G'(x) = \sin(x^3)^2 \cdot 3x^2$$

$$H(x) = \int_{x^2}^{x^3} \sin t^2 \, dt = \int_0^{x^3} \sin t^2 \, dt - \int_0^{x^2} \sin t^2 \, dt = F(x^3) - F(x^2) =$$

$$\Rightarrow H'(x) = \sin x^6 \cdot 3x^2 - \sin x^4 \cdot 2x$$

Integrálfv

$f: [a,b] \to \mathbb{R}$ R-ильható

$F: [a,b] \to \mathbb{R}$ $F(x) := \int_a^x f(t)\,dt$ → folytonos

→ ha f folytonos x_0-ban

↪ $F'(x_0) = f(x_0)$ $(\exists)$

(1) Adjuk meg azt a minimális fokszámú $P: \mathbb{R} \to \mathbb{R}$ polinomot, melynek a 2-ben minimuma, 6-ban maximuma van és $P(2)=0$, $P(6)=32$!

$\Rightarrow \exists\, a \in \mathbb{R}: \quad P'(x) = a(x-2)(x-6) \quad \forall x \in \mathbb{R}$

$$P(x) = \int_2^x P'(t)\,dt \underset{P(2)=0}{=} a\left(\frac{x^3}{3} - 4x^2 + 12x - \frac{32}{3}\right)$$

$P(6) = 32$

↪ $P(x) = -x^3 + 12x^2 - 36x + 32$

(2) $f: \mathbb{R} \to \mathbb{R}$ $f(x+p) = f(x)$ $p \in \mathbb{R}^+$

$$F(x) := \int_{x_0}^x f(t)\,dt$$

Biz be, h $\exists$ $G: \mathbb{R} \to \mathbb{R}$ $G(x+p) = G(x)$, hogy $F(x) = G(x) + c$

$$F(x+p) - F(x_0+p) = \int_{x_0}^{x+p} f(x)\,dx - \int_{x_0}^{x_0+p} f(x)\,dx = \int_{x_0+p}^{x+p} f(x)\,dx =$$

$$= \int_{x_0}^{x} f(x)\,dx = F(x)$$

$\uparrow$

f periodic

$$F(x_0+p) = \int_{x_0}^{x_0+p} f(x)\,dx = \int_0^p f(x)\,dx$$

$$\hookrightarrow \quad F(x+p) - F(x) = F(x_0+p) = \int_0^p f(x)\,dx =: A$$

- ha $A = 0 \Rightarrow \checkmark$
- ha $A \neq 0$ $\quad G(x) := F(x) - \frac{A}{p}x$

$$G(x+p) = F(x+p) - \frac{A}{p}(x+p) = \underbrace{F(x+p) - A}_{F(x)} - \frac{A}{p}x = G(x) \checkmark$$

Sorozatok határértéke:

(1) $$a_n := n\left(\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots + \frac{1}{(2n)^2}\right)$$

$\lim a_n = ?$

$$a_n := \frac{1}{n}\left(\frac{1}{\left(1+\frac{1}{n}\right)^2} + \frac{1}{\left(1+\frac{2}{n}\right)^2} + \dots + \frac{1}{\left(1+\frac{n}{n}\right)^2}\right)$$

$f : [0,1] \to \mathbb{R}$ $\qquad f(x) := \dfrac{1}{(1+x)^2}$

beosztása $[0,1]$-nek $\qquad D_n = \left\{ \frac{i}{n} : i = 0,1,\dots,n \right\}$ $\qquad n = 1,2,\dots$

f R-integrálható $\qquad D_n$ finomodó felosztás $\qquad a_n = S(f, D_n)$

D_n felosztáshoz tartozó [illegible]

$f \searrow$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} S(f, D_n) = \int_0^1 \frac{1}{(1+x)^2}\,dx = \frac{1}{2}$$

(2) $$a_n := \frac{\pi}{n}\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \dots + \sin\frac{(n-1)\pi}{n}\right) \qquad \lim a_n = ?$$

$$a_n = \frac{\pi}{n}\left(\sin 0 + \sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \dots + \sin\frac{n-1}{n}\pi\right)$$

$f : [0,\pi] \to \mathbb{R}$ $\qquad f(x) = \sin x$

$$\lim_{n\to\infty} a_n = \int_0^\pi \sin x\,dx = 2$$

$$F(x+p)-F(x_0+p) = \int_{x_0}^{x+p} f(x)\,dx - \int_{x_0}^{x_0+p} f(x)\,dx = \int_{x_0+p}^{x+p} f(x)\,dx =$$

$$\underset{\substack{\uparrow \\ f \text{ periodikus}}}{=} \int_{x_0}^{x} f(x)\,dx = F(x)$$

$$F(x_0+p) = \int_{x_0}^{x_0+p} f(x)\,dx = \int_0^p f(x)\,dx$$

$$\hookrightarrow \quad F(x+p)-F(x) = F(x_0+p) = \int_0^p f(x)\,dx =: A$$

- ha $A=0 \Rightarrow \checkmark$
- ha $A \neq 0$ $\qquad G(x) := F(x) - \frac{A}{p}x$

$$G(x+p) = F(x+p) - \frac{A}{p}(x+p) = \underbrace{F(x+p)-A}_{F(x)} - \frac{A}{p}x = G(x) \checkmark$$

(3)

$$a_n = \frac{1^\alpha + 2^\alpha + \dots + n^\alpha}{n^{\alpha+1}} \qquad \alpha \in \mathbb{R}$$

$$a_n = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^\alpha$$

$$\lim_{n\to\infty} a_n = \int_0^1 x^\alpha \, dx = \frac{1}{\alpha+1} \quad !$$

(4) $a \in \mathbb{R}^+$ Bil ke

$$\log x - \log a = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-a)^n}{n\, a^n} \qquad x \in (0, 2a)$$

$x \in (0, 2a)$

$$\frac{1}{x} = \frac{1}{a} \, \frac{1}{1 + \frac{x-a}{a}} = \frac{1}{a} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x-a}{a}\right)^n$$

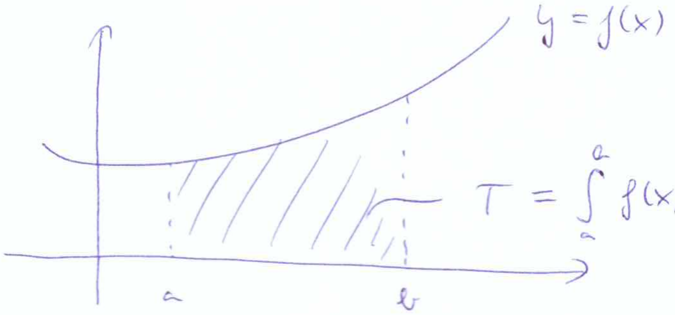
$$\int_a^x \frac{1}{t} \, dt = \ln x - \ln a = \sum_{n=0}^{\infty} \int_a^x \frac{(-1)^n}{a^{n+1}} (t-a)^n \, dt =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x-a)^{n+1}}{(n+1)\, a^{n+1}} \quad ✓$$

1/

Területszámítás

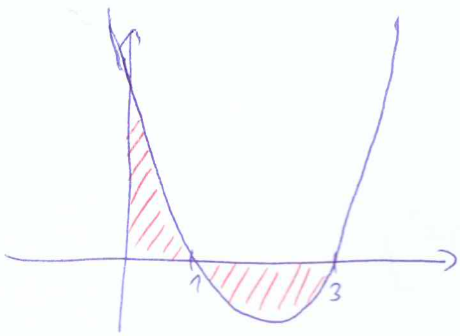
(1)



$$T = \int_a^b f(x)\,dx \quad , \text{ha } f(x) \geq 0$$

Pl1 Mennyi az $y = x^2 - 4x + 3$ függvény görbéje és a tengelyek által körbezárt síkrész területe?

$$y = x^2 - 4x + 3 = (x-1)(x-3)$$

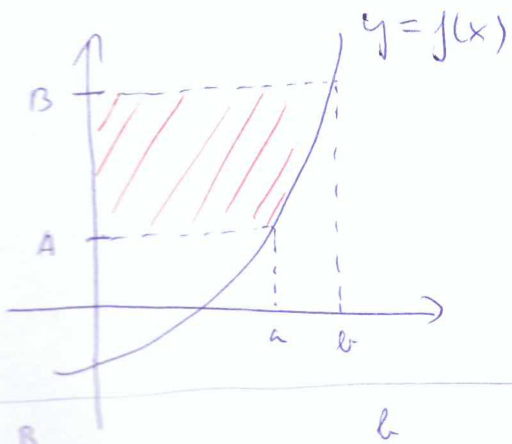


$$T = \int_0^1 (x^2-4x+3)\,dx - \int_1^3 (x^2-4x+3)\,dx = \left[\frac{x^3}{3} - 2x^2 + 3x\right]_0^1 - \left[\frac{x^3}{3} - 2x^2 + 3x\right]_1^3 =$$

$$\left(\frac{1}{3} - 2 + 3\right) - \left((9 - 18 + 9) - \left(\frac{1}{3} - 2 + 3\right)\right) = \frac{8}{3}$$

vagy $T = \left| \int_0^1 (x^2-4x+3)\,dx \right| + \left| \int_1^3 (x^2-4x+3)\,dx \right|$

(2)



$f(a) = A \rightarrow a = f^{-1}(A)$

$f(b) = B \rightarrow b = f^{-1}(B)$

$y = y(x) \rightarrow x = x(y)$ ha invertálható

$A \leq f(x) \leq B$

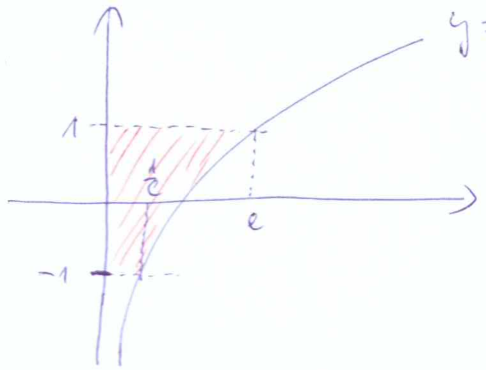
$f^{-1}(A) \leq x \leq f^{-1}(B)$

$a \leq x \leq b$

$$\boxed{T = \int_A^B x(y)\,dy = \int_a^b x\, f'(x)\,dx}$$

$y := f(x)$ helyettesítés $\quad y' = \frac{dy}{dx} = f'(x) \rightsquigarrow dy = f'(x)\,dx$

Pl

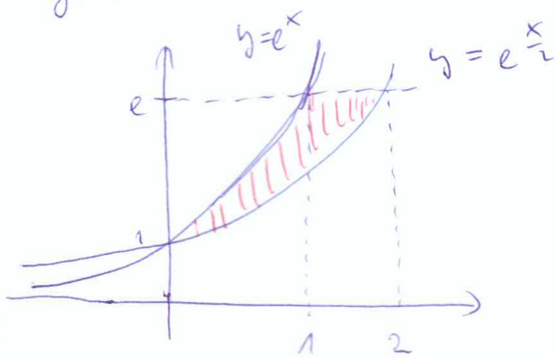


$y = \ln x \quad \rightsquigarrow \quad x = e^y$

$$T = \int_{-1}^{1} e^y \, dy = \left[e^y \right]_{-1}^{1} = e - \frac{1}{e}$$

vagy
$$T = \int_{\frac{1}{e}}^{e} x (\ln x)' \, dx = \int_{\frac{1}{e}}^{e} dx = \left[x \right]_{\frac{1}{e}}^{e} = e - \frac{1}{e}$$

③ görbék közti terület



$$T = \int_0^2 x \left(e^{\frac{x}{2}}\right)' dx - \int_0^1 x (e^x)' dx =$$
$$= \left[x e^{\frac{x}{2}} \right]_0^2 - \int_0^2 e^{\frac{x}{2}} dx - \left[x e^x \right]_0^1 + \int_0^1 e^x dx =$$
$$= 2e - \left[2 e^{\frac{x}{2}} \right]_0^2 - e + \left[e^x \right]_0^1 =$$
$$= 2e - 2e + 2 - e + e - 1 = \underline{\underline{1}}$$

$$\cancel{T = \int_1^2 \left(x \left(e^{\frac{x}{2}}\right)' - x (e^x)' \right) dx = \int_1^2 x \left(\frac{1}{2} e^{\frac{x}{2}} - e^x \right) dx} =$$

$$\cancel{u = x \to u' = 1}$$
$$\cancel{v' = \frac{1}{2} e^{\frac{x}{2}} - e^x \to v = e^{\frac{x}{2}} - e^x}$$

$$\cancel{= \left[x \left(e^{x/2} - e^x \right) \right]_1^2 - \underbrace{\int_1^2 \left(e^{x/2} - e^x \right) dx}_{\left[2 e^{x/2} - e^x \right]_1^2} = \left[2(e - e^2) - (e^{1/2} - e) \right] - \left[2e - e^2 - (2e^{1/2} - e) \right]}$$
$$\cancel{= 3e - 2e^2 - e^{1/2} + e^2 - 3e + 2e^{1/2} =}$$
$$\cancel{= e^{1/2} - e^2}$$

2/

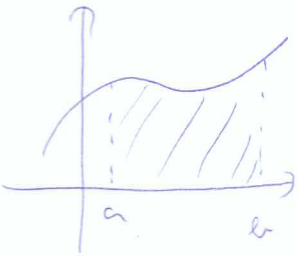
④ paraméteres megadás

$x = x(t)$ $\quad y = y(t)$ $\qquad \alpha \le t \le \beta \quad \rightsquigarrow \quad a := x(\alpha),\ b := x(\beta) \Rightarrow \boxed{a \le x \le b}$

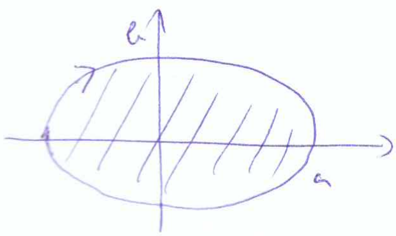
$T = \int dT \rightsquigarrow dT = y\,dx = y(t)\,\dot{x}(t)\,dt$

$$\boxed{T = \int_a^b y\,d(x) = \int_\alpha^\beta y(t)\cdot\dot{x}(t)\,dt}$$

$y\,dx = y(t)\,\dot{x}(t)\,dt$



Pl/ ellipszis területe



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$x(t) = a\cos t$ $\quad y(t) = b\sin t$ $\qquad 0 \le t \le 2\pi \quad \to \quad \dot{x}(t) = -a\sin t$

$$\frac{T}{2} = \int_\pi^0 b\sin t\,(-a\sin t)\,dt = -ab\int_\pi^0 \sin^2 t\,dt = +\frac{ab}{2}\int_0^\pi (1-\cos 2t)\,dt =$$

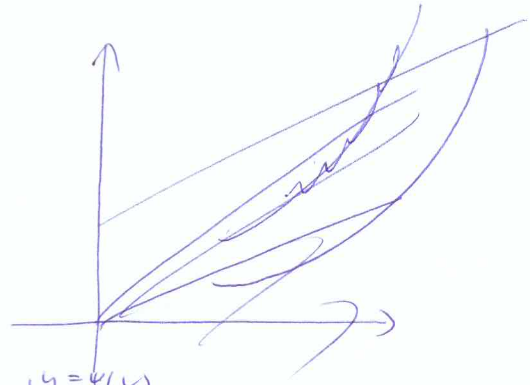
$$= +\frac{ab}{2}\left[t - \frac{\sin 2t}{2}\right]_0^\pi = \frac{ab\pi}{2}$$

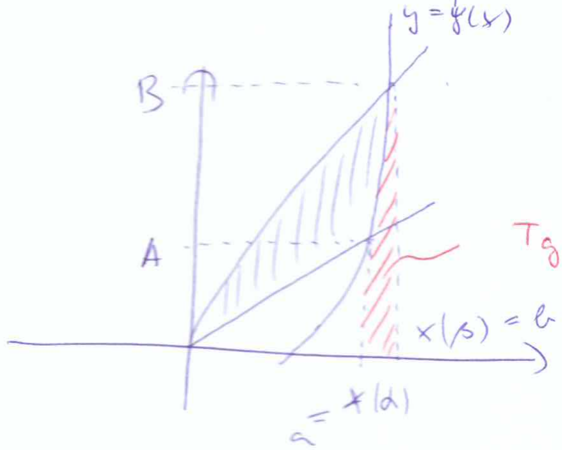
$\Downarrow$

$$\boxed{T = ab\pi}$$

Szektorterület

① $s:\ x = x(t)$
$\quad\ \ y = y(t)$





$a \le x(t) \le b$

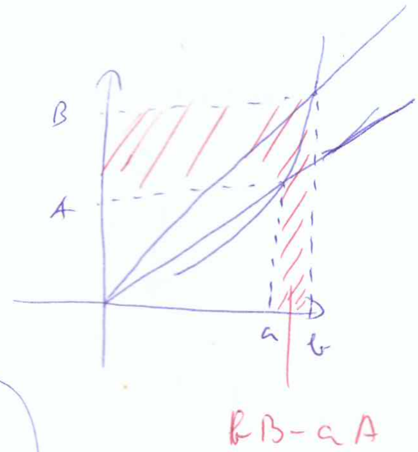
T_g : görbe alatti terület

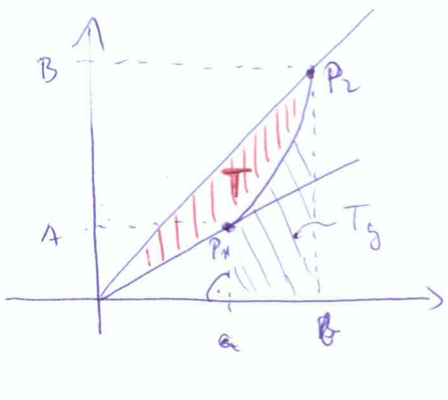
$$T_g = \int_a^b y\,dx = \int_\alpha^\beta y(t)\,\dot{x}(t)\,dt \underset{\text{parc. int}}{=} \big[y(t)x(t)\big]_\alpha^\beta - \int_\alpha^\beta \dot{y}(t)\,x(t)\,dt$$

tehát

$$T_g = \int_\alpha^\beta y(t)\,\dot{x}(t)\,dt$$

$$T_g = \underbrace{\big[y(t)x(t)\big]_\alpha^\beta}_{y(\beta)x(\beta) - y(\alpha)x(\alpha) \,=\, bB - aA} - \int_\alpha^\beta \dot{y}(t)\,x(t)\,dt$$





$$\Rightarrow\quad T + T_g + \frac{a\cdot A}{2} = \frac{b\,B}{2}$$

$$\hookrightarrow\quad T = \frac{b\,B}{2} - \frac{a\,A}{2} - T_g = \frac{1}{2}\int_\alpha^\beta \big(x(t)\dot{y}(t) - \dot{x}(t)y(t)\big)\,dt$$

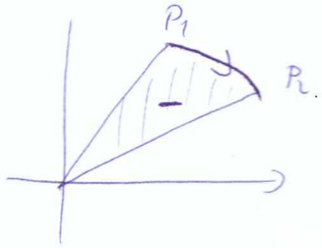
3/

szektorterület: $x = x(t)$, $y = y(t)$ $\quad \left.\begin{array}{l} x(\alpha)=a \\ x(\beta)=b \end{array}\right\}$ korlátai

$$T = \frac{1}{2}\int_{\alpha}^{\beta} \left(x(t)\dot{y}(t) - \dot{x}(t)y(t)\right)dt$$

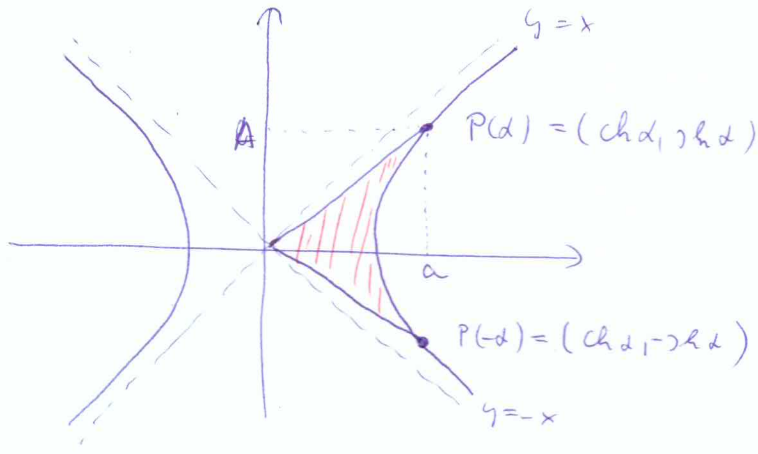
Megj: előjeles terület: $T > 0$, ha $P_1 \to P_2$ pozitív irányú

$T < 0$, ha $P_1 \to P_2$ negatív —ıı—



Pl.

① $\left.\begin{array}{l} x = \mathrm{ch}\, t \\ y = \mathrm{sh}\, t \end{array}\right\}$ egységnyi hiperbola $-\alpha \le t \le \alpha$ szektorának területe?



$x^2 - y^2 = 1$

$$\Rightarrow \quad x(t)\dot{y}(t) - \dot{x}(t)y(t) = \mathrm{ch}^2 t - \mathrm{sh}^2 t = 1.$$

$$\hookrightarrow \quad T = \frac{1}{2}\int_{-\alpha}^{\alpha} 1\,dt = \frac{1}{2}\left[t\right]_{-\alpha}^{\alpha} = \underline{\underline{\alpha}}$$

vagyis $T = \alpha \underset{\substack{\uparrow \\ a = \mathrm{ch}\,\alpha}}{=} \mathrm{arch}\, a \underset{\substack{\uparrow \\ A = \mathrm{sh}\,\alpha}}{=} \mathrm{arsh}\, A \quad \Rightarrow$ area elnevezés oka

② polárkoordinátásan adott

$\boxed{r = r(\varphi)}$

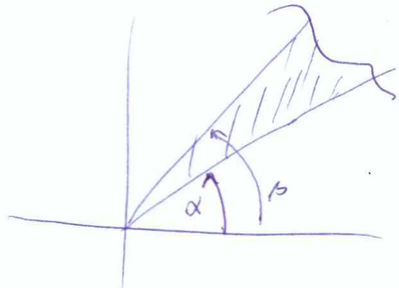
$$\left.\begin{aligned} x(\varphi) &= r(\varphi)\cos\varphi \\ y(\varphi) &= r(\varphi)\sin\varphi \end{aligned}\right\}$$

$\alpha \le \varphi \le \beta$

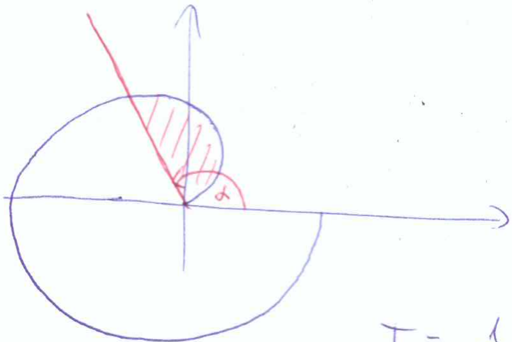
$\Downarrow$

$$x\dot{y} - \dot{x}y = r\cos\varphi(\dot{r}\sin\varphi + r\cos\varphi) - (\dot{r}\cos\varphi - r\sin\varphi)r\sin\varphi =$$
$$= r^2$$

$\hookrightarrow$ $$\boxed{T = \frac{1}{2}\int_\alpha^\beta r^2(\varphi)\,d\varphi}$$



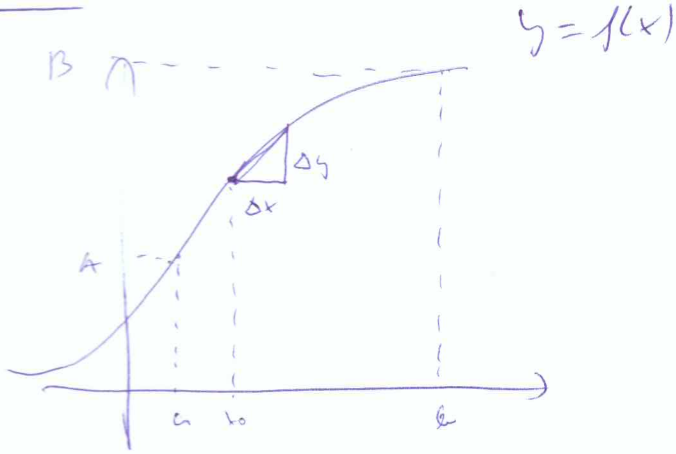
Pl $r = c\varphi$ Arkhimédeszi-spirális

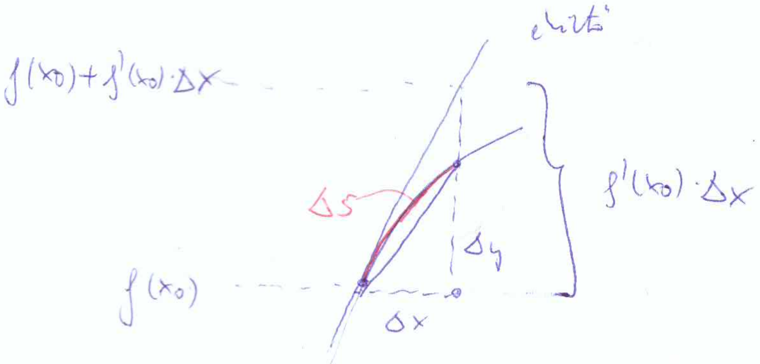


$(0,\alpha)$ közé eső szektorterület

$$T = \frac{1}{2}\int_0^\alpha c^2\varphi^2\,d\varphi = \frac{c^2}{2}\left[\frac{\varphi^3}{3}\right]_0^\alpha = \underline{\underline{\frac{c^2\alpha^3}{6}}}$$

4) <u>ívhossz</u>





$$y = f(x_0) + f'(x_0)(x-x_0)$$

$$\sqrt{(\Delta x)^2+(\Delta y)^2} = \sqrt{1+\frac{\Delta y^2}{\Delta x^2}}\cdot\Delta x \leq \Delta s \leq \sqrt{\Delta x^2 + f'(x)^2\,\Delta x^2} =$$

$$= \Delta x\sqrt{1+f'(x)^2}$$

$$\downarrow \cancel{\Delta x = dx \to 0}$$

$$\sqrt{1+\frac{\Delta y^2}{\Delta x^2}} \leq \frac{\Delta s}{\Delta x} \leq \sqrt{1+f'(x)^2}$$

$$\downarrow \Delta x \to 0$$

$$\lim_{\Delta x\to 0}\frac{\Delta s}{\Delta x} = \frac{ds}{dx} = \sqrt{1+[f'(x)]^2}$$

$$ds = \sqrt{1+[f'(x)]^2}\,dx$$

$$\Rightarrow \boxed{s = \int_a^b \sqrt{1+[f'(x)]^2}\,dx}$$

Pl

$y = \operatorname{ch} x$ ívhossza $a \le x \le b$ -n

$$S = \int_a^b \sqrt{1+\operatorname{sh}^2 x}\, dx = \int_a^b \operatorname{ch} x\, dx = \operatorname{sh} b - \operatorname{sh} a$$

$$\uparrow$$

$$(\operatorname{ch} x)' = \operatorname{sh} x$$

(2) paraméteres esetben $\quad x = x(t) \quad y = y(t) \qquad \alpha \le t \le \beta$

$$ds = \sqrt{dx^2+dy^2} = \sqrt{\frac{dx^2}{dt^2}+\left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{\dot{x}(t)^2+\dot{y}(t)^2}\, dt$$

$$\hookrightarrow \quad S = \int_\alpha^\beta \sqrt{\dot{x}^2(t)+\dot{y}^2(t)}\, dt$$

Pl

$$\left.\begin{aligned} x &= r(t-\sin t) \\ y &= r(1-\cos t) \end{aligned}\right\} \text{ ciklois egy ívének hossza}$$

$$\hookrightarrow \quad \left.\begin{aligned} \dot{x} &= r(1-\cos t) \\ \dot{y} &= r\sin t \end{aligned}\right\} \quad \dot{x}^2+\dot{y}^2 = r^2(1-\cos t)^2 + r^2\sin^2 t =$$

$$= r^2\left[1-2\cos t+\cos^2 t+\sin^2 t\right] =$$

$$= 2r^2(1-\cos t) = 4r^2\sin^2\frac{t}{2}$$

$$\cancel{S = \int_0^{2\pi} 2r^2(1-\cos t)\, dt = 2r^2\left[t-\sin t\right]_0^{2\pi} = 4\pi r}$$

$$S = \int_0^{2\pi} 2r\sin\frac{t}{2}\, dt = 2r\left[-2\cos\frac{t}{2}\right]_0^{2\pi} = -4r\left[-1-1\right] = \underline{\underline{8r}}$$

5)

(3) polarkoordinates alak

$$r = r(\varphi) \qquad \alpha \le \varphi \le \beta$$

$$\Updownarrow$$

$$x(\varphi) = r(\varphi)\cos\varphi$$
$$y(\varphi) = r(\varphi)\sin\varphi$$

$\hookrightarrow$ $\dot{x}(\varphi) = \dot{r}(\varphi)\cos\varphi - r(\varphi)\sin\varphi$

$\dot{y}(\varphi) = \dot{r}(\varphi)\sin\varphi + r(\varphi)\cos\varphi$

$$\sqrt{\dot{x}^2+\dot{y}^2} = \sqrt{\dot{r}^2(\varphi)+r^2(\varphi)}$$

$$s = \int_\alpha^\beta \sqrt{\dot{r}^2(\varphi)+r^2(\varphi)}\,d\varphi$$

Pel $\quad r = a\varphi \qquad 0 \le \varphi \le \alpha \quad$ Arkhimedeszi spirális

$$\Downarrow$$

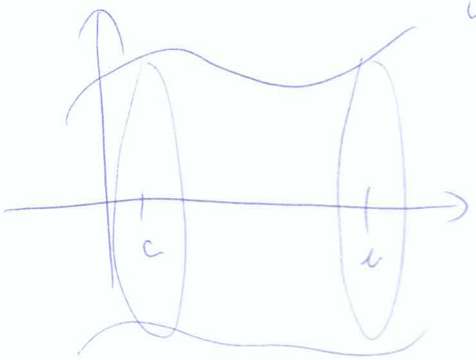
$$\dot{r} = a$$

$$s = \int_0^\alpha \sqrt{a^2+a^2\varphi^2}\,d\varphi = a\int_0^\alpha \sqrt{1+\varphi^2}\,d\varphi =$$

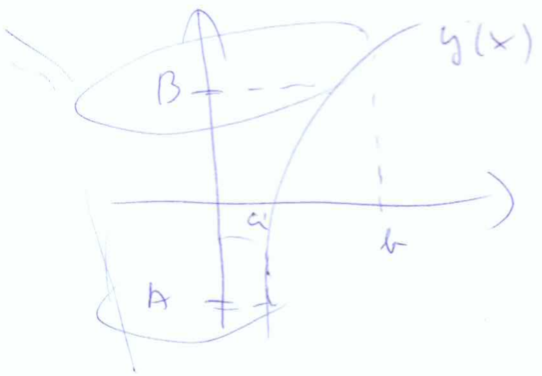
$\varphi = \operatorname{sh} t \qquad d\varphi = \operatorname{ch} t\,dt \qquad t = \operatorname{arsh}\varphi$

$$\ldots = \frac{a}{2}\left[\alpha\sqrt{1+\alpha^2} + \ln\left(\alpha+\sqrt{1+\alpha^2}\right)\right]$$

Forgástest térfogata



$$\boxed{V_x(a,b) = \pi \int_a^b y(x)^2 \, dx}$$



$\rightsquigarrow \quad x = x(y)$

$$V_y(A,B) = \pi \int_A^B x^2(y) \, dy$$

~~$x(a) = A \quad x(b)$~~

$y(a) = A \qquad y(b) = B$

$y = y(x) \rightsquigarrow dy = y' dx$

$A \le y \le B \implies a \le x \le b$

$$\hookrightarrow \boxed{V_y(a,b) = \pi \int_a^b x^2 y'(x) \, dx}$$

paraméteres megadás:

$$V_x(\alpha,\beta) = \pi \int_\alpha^\beta y^2(t) \, \dot{x}(t) \, dt$$