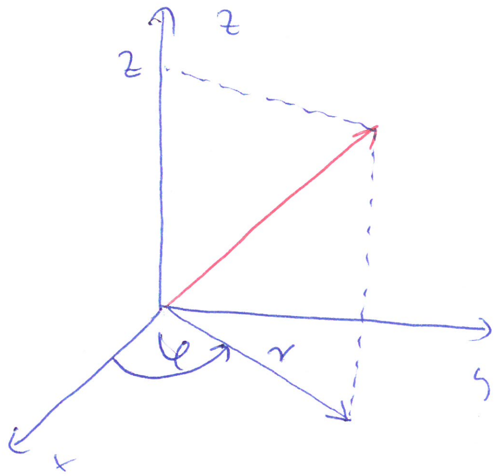
Térfogati integrálok (hármasintegrálok) transzformációja

Hengerkoordináták (térbeli polárkoordináták)



$$\left.\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \\ z &= z \end{aligned}\right\} \quad g(r,\varphi,z) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix}$$

$$\Updownarrow$$

$$r = \sqrt{x^2+y^2}$$

$$\varphi = \operatorname{arctg}\frac{y}{x}$$

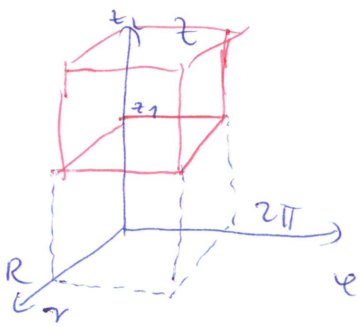
$$z = z$$

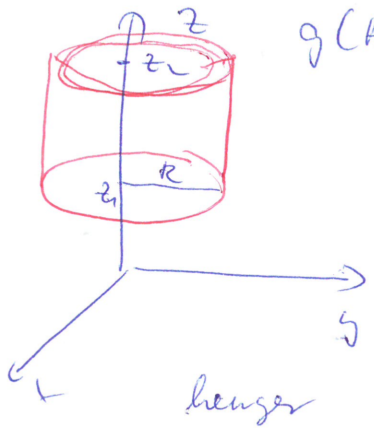
Jacobi-determináns:

$$\det g'(r,\varphi,z) = \left|\frac{\partial(x,y,z)}{\partial(r,\varphi,z)}\right| = \det\begin{pmatrix} x'_r & x'_\varphi & x'_z \\ y'_r & y'_\varphi & y'_z \\ z'_r & z'_\varphi & z'_z \end{pmatrix} =$$

$$= \det\begin{pmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underline{\underline{r}}$$

$$A:\quad \begin{aligned} 0 &\le r \le R \\ 0 &\le \varphi \le 2\pi \\ z_1 &\le z \le z_2 \end{aligned} \qquad \xrightarrow{\quad g \quad} \qquad g(A)$$

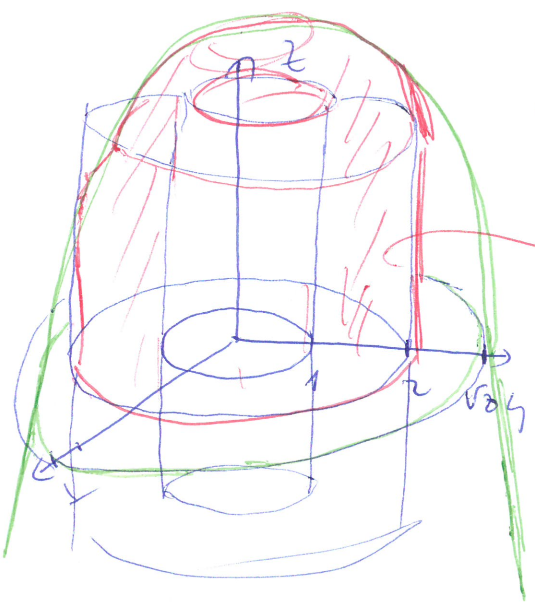




358/

Példa ① $V := \{(x,y,z) \in \mathbb{R}^3 : 1 \leq x^2+y^2 \leq 4,\ 0 \leq z \leq 8-x^2-y^2\}$

$$I := \int_V x^2\,dV = ?$$



$z = 8-x^2-y^2$: forgási paraboloid

V: a két henger által a só határolja „oldalról", felülről a paraboloid, alulról az (x,y) sík.

forgásszimmetrikus $\rightsquigarrow$ hengerkoordináták: $x = r\cos\varphi$, $y = r\sin\varphi$, $z = z$

V: $1 \leq r^2 \leq 4 \rightsquigarrow 1 \leq r \leq 2$

forgástest $\rightsquigarrow$ $0 \leq \varphi \leq 2\pi$

$0 \leq z \leq 8-x^2-y^2 = 8-r^2$

$\Rightarrow$ $g^{-1}(V)$:
$$\left.\begin{array}{l} 1 \leq r \leq 2 \\ 0 \leq \varphi \leq 2\pi \\ 0 \leq z \leq 8-r^2 \end{array}\right\}$$

$$I = \int_V x^2\,dV = \underbrace{\int_1^2\int_0^{2\pi}\int_0^{8-r^2}}_{g^{-1}(V)} r^2\cos^2\varphi \cdot r\,dz\,d\varphi\,dr \;=$$

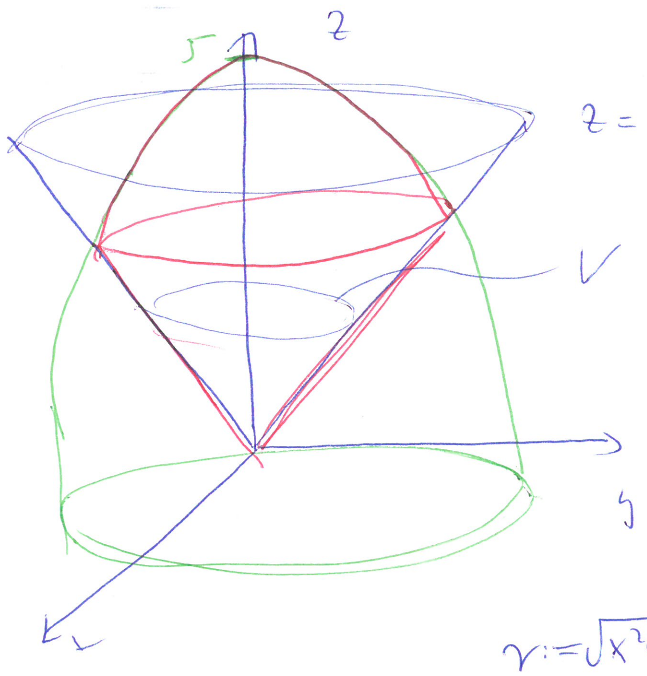
Jacobi-det!

359)

$$\text{Ⓔ} = \int_1^2 \int_0^{2\pi} \underbrace{\int_0^{8-r^2} r^3 \cos^2\varphi \, dz}_{\left[r^3\cos^2\varphi\, z\right]_{z=0}^{8-r^2} = (8r^3 - r^5)\cos^2\varphi} \, d\varphi \, dr =$$

$$= \underbrace{\int_1^2 (8r^3 - r^5)\, dr}_{\left[\frac{8r^4}{4} - \frac{r^6}{6}\right]_1^2 = \frac{39}{2}} \cdot \underbrace{\int_0^{2\pi} \cos^2\varphi \, d\varphi}_{\frac{1}{2}\int_0^{2\pi}(1+\cos 2\varphi)\,d\varphi = \frac{1}{2}\left[\varphi + \frac{\sin 2\varphi}{2}\right]_0^{2\pi} = \pi} = \frac{39}{2}\pi$$

② Határozzuk meg a $z = 5 - x^2 - y^2$ és a $z = \sqrt{x^2+y^2}$ felületek közé zárt korlátos test térfogatát.



- $z = \sqrt{x^2+y^2} \geq 0$ forgáskúp
- $z = 5 - x^2 - y^2$ forgási paraboloid

metszetgörbe:

$$\sqrt{x^2+y^2} = 5 - x^2 - y^2$$

$r := \sqrt{x^2+y^2}$

$$r = 5 - r^2$$

$$\Leftrightarrow r^2 + r - 5 = 0$$

⇝ „fagyi" térfogata?

360)

$$r^2+r-5=0 \rightsquigarrow D=1+4\cdot 5=21 \quad \text{(diszkrimináns)}$$

$$r_{1,2}=\frac{-1\pm\sqrt{21}}{2} = \begin{cases} -\frac{\sqrt{21}+1}{2}<0 : \lightning \\ \frac{\sqrt{21}-1}{2} \end{cases}$$

forgástest $\rightsquigarrow$ hengerkoordináták:

$$\sqrt{x^2+y^2}\le z\le 5-x^2-y^2 \rightsquigarrow \left.\begin{array}{l} \bullet\ r\le z\le 5-r^2 \\ \bullet\ 0\le r\le \frac{\sqrt{21}-1}{2} \\ \bullet\ 0\le\varphi\le 2\pi \end{array}\right\} A$$

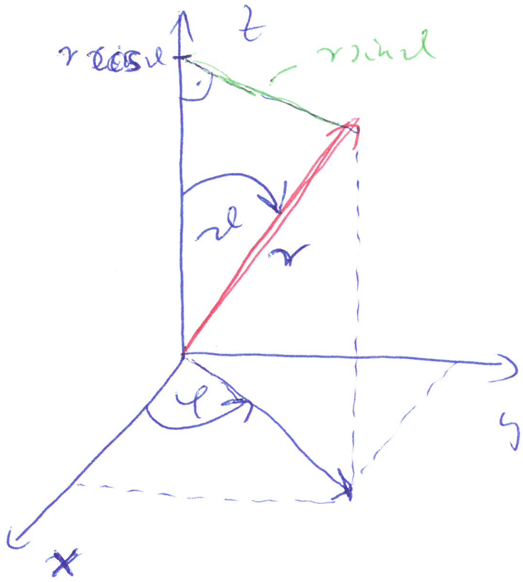
$$g(A)=V$$

$$\int_V dV=\int_0^{2\pi}\int_0^{\frac{\sqrt{21}-1}{2}}\underbrace{\int_r^{5-r^2} \underset{\uparrow\,\text{Jacobi-det}}{r}\,dz}_{\left[rz\right]_{z=r}^{5-r^2}=5r-r^3-r^2}\,dr\,d\varphi \;=$$

$$=\underbrace{\int_0^{2\pi}d\varphi}_{2\pi}\cdot\underbrace{\int_0^{\frac{\sqrt{21}-1}{2}}(5r-r^3-r^2)\,dr}_{\left[5\frac{r^2}{2}-\frac{r^4}{4}-\frac{r^3}{3}\right]_0^{\frac{\sqrt{21}-1}{2}}} = \ldots$$

361)

Gömbi koordináták (r, ϑ, φ)



- $r \geq 0$ ~> sugár
- $0 \leq \vartheta \leq \pi$ ~> azimutszög
- $0 \leq \varphi < 2\pi$ ~> polárszög

$$\left.\begin{aligned} x &= r \sin\vartheta \cos\varphi \\ y &= r \sin\vartheta \sin\varphi \\ z &= r \cos\vartheta \end{aligned}\right\}$$

$$g(r,\vartheta,\varphi) = \begin{pmatrix} r\sin\vartheta\cos\varphi \\ r\sin\vartheta\sin\varphi \\ r\cos\vartheta \end{pmatrix}$$

Jacobi-det:

$$\det g'(r,\vartheta,\varphi) = \left| \frac{\partial(x,y,z)}{\partial(r,\vartheta,\varphi)} \right| = \det \begin{pmatrix} x'_r & x'_\vartheta & x'_\varphi \\ y'_r & y'_\vartheta & y'_\varphi \\ z'_r & z'_\vartheta & z'_\varphi \end{pmatrix}$$

$$= \det \begin{pmatrix} \sin\vartheta\cos\varphi & r\cos\vartheta\cos\varphi & -r\sin\vartheta\sin\varphi \\ \sin\vartheta\sin\varphi & r\cos\vartheta\sin\varphi & r\sin\vartheta\cos\varphi \\ \cos\vartheta & -r\sin\vartheta & 0 \end{pmatrix} =$$

$$= \sin\vartheta\cos\varphi \cdot r^2\sin^2\vartheta\cos\varphi + r\cos\vartheta\cos\varphi\, r\sin\vartheta\cos\vartheta\cos\varphi \; -$$

$$- r\sin\vartheta\sin\varphi \underbrace{\left(-r\sin^2\vartheta\sin\varphi - r\cos^2\vartheta\sin\varphi\right)}_{-r\sin\varphi(\sin^2\vartheta+\cos^2\vartheta) = -r\sin\varphi} \quad =$$

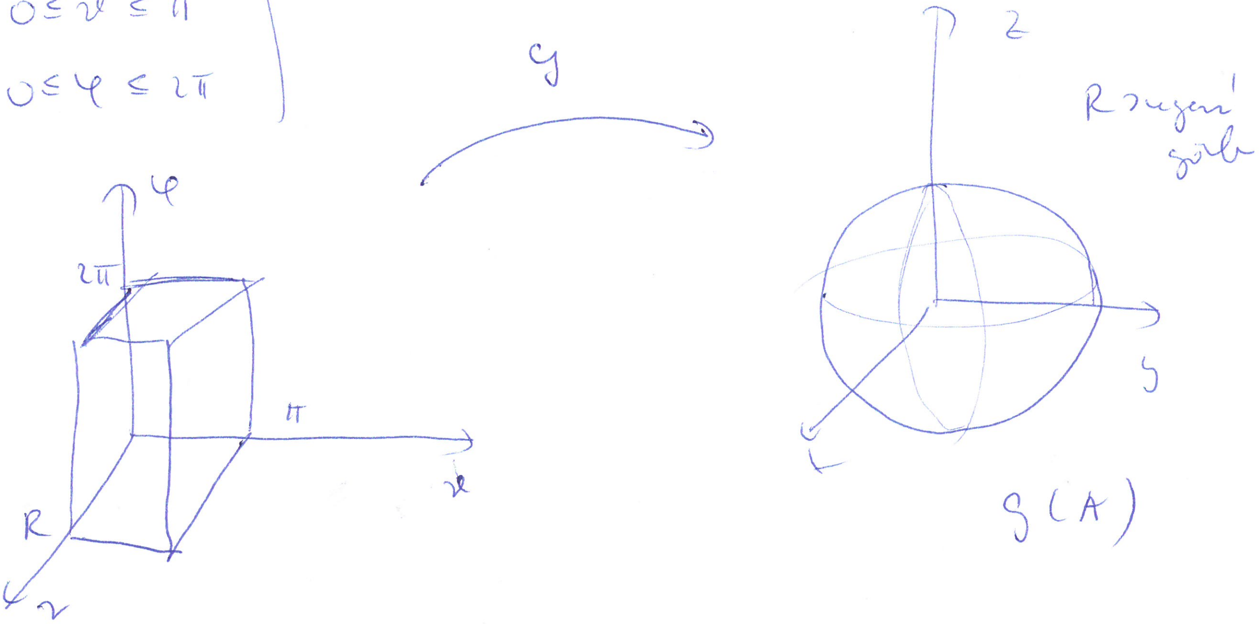
362/

$$\Leftarrow \quad \underbrace{r^2 \sin^3\vartheta \cos^2\varphi + r^2 \sin\vartheta \cos^2\vartheta \cdot \cos^2\varphi}_{r^2 \sin\vartheta \cos^2\varphi\, (\underbrace{\sin^2\vartheta + \cos^2\vartheta}_{=1})} + r^2 \sin\vartheta \sin^2\varphi =$$

$$= r^2 \sin\vartheta\, (\underbrace{\cos^2\varphi + \sin^2\varphi}_{=1}) = r^2 \sin\vartheta$$

$$\boxed{\left| \frac{\partial(x,y,z)}{\partial(r,\vartheta,\varphi)} \right| = r^2 \sin\vartheta}$$

$$A: \quad \left.\begin{array}{l} 0 \le r \le R \\ 0 \le \vartheta \le \pi \\ 0 \le \varphi \le 2\pi \end{array}\right\}$$

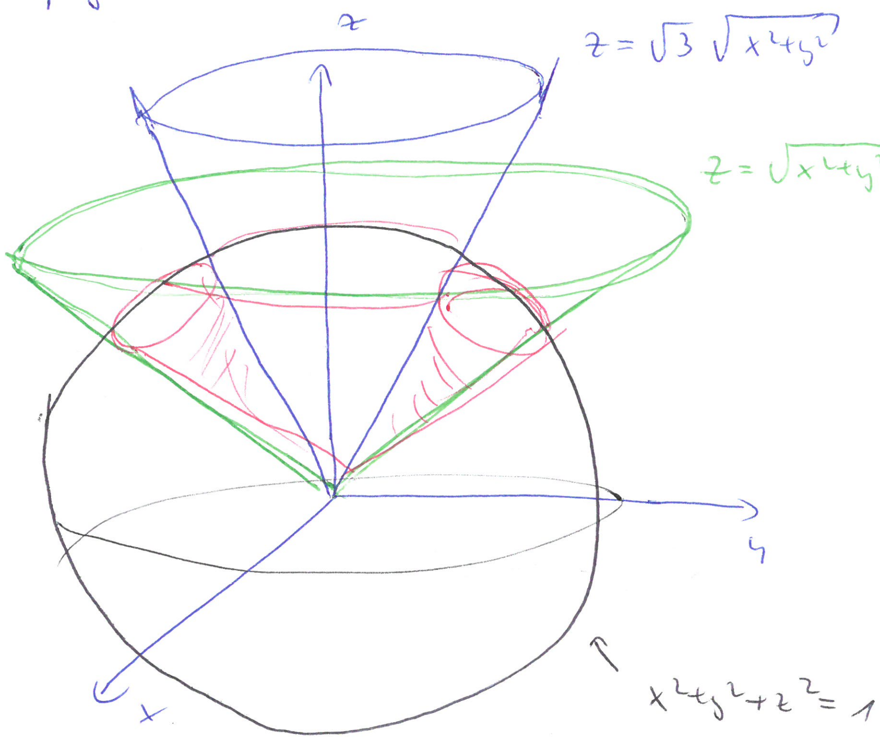


363)

Példák

(1) $V := \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2\le 1 ,\ \sqrt{x^2+y^2}\le z\le \sqrt{3}\sqrt{x^2+y^2}\}$

V térfogata?



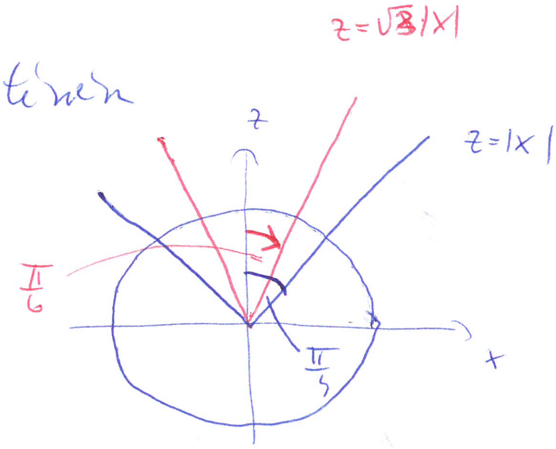
- $z=\sqrt{x^2+y^2}$

  ↳ $2\cdot\frac{\pi}{4}$ nyílásszögű kúp

- $z=\sqrt{3}\sqrt{x^2+y^2}$

  ↳ $2\cdot\frac{\pi}{6}$ nyílásszögű kúp

V: a két kúp és a gömb közé eső térrész



- $z=\sqrt{x^2+y^2}$ kúpnál $\vartheta=\frac{\pi}{4}$
- $z=\sqrt{3}\sqrt{x^2+y^2}$ kúpnál $\vartheta=\frac{\pi}{6}$

↳ gömbi koordináták: $0\le r\le 1$

$$\frac{\pi}{6}\le\vartheta\le\frac{\pi}{4}$$

$$0\le\varphi\le 2\pi$$

364)

Jacobi-det

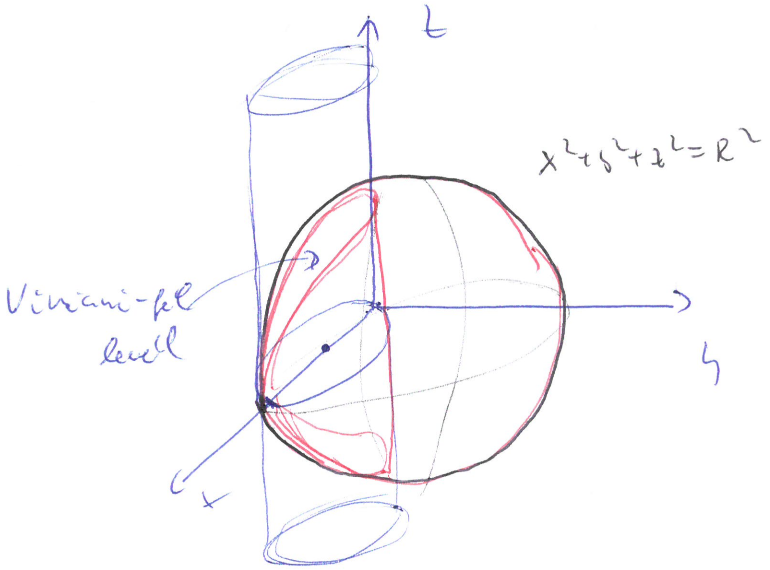
$$\Rightarrow m_3(V) = \int_V 1\, dV = \int_0^{2\pi}\int_0^{1}\int_{\pi/6}^{\pi/4} r^2 \sin\vartheta \, d\vartheta\, dr\, d\varphi =$$

$$= \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \cdot \underbrace{\int_0^{1} r^2\, dr}_{\left[\frac{r^3}{3}\right]_0^1 = \frac{1}{3}} \cdot \underbrace{\int_{\pi/6}^{\pi/4} \sin\vartheta\, d\vartheta}_{\left[-\cos\vartheta\right]_{\pi/6}^{\pi/4} = \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}} = \underline{\underline{\frac{2\pi}{3}\,\frac{\sqrt{3}-\sqrt{2}}{2}}}$$

② Viviani-féle test térfogata:

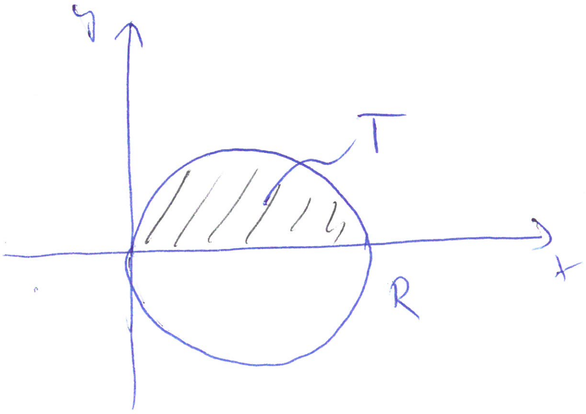
$x^2+y^2+z^2 \le R^2$ gömbből az $x^2-Rx+y^2=0$ henger által kivágott test térfogata?

$$x^2 - Rx + y^2 = 0 \iff \left(x - \frac{R}{2}\right)^2 + y^2 = \frac{R^2}{4}$$



Mivel V szimmetrikus az xy síkra ~~számoljuk~~ és az xz síkra, számoljuk ki a térfogat negyedét!

365/



~> itt megoldható kettős integrállal is:

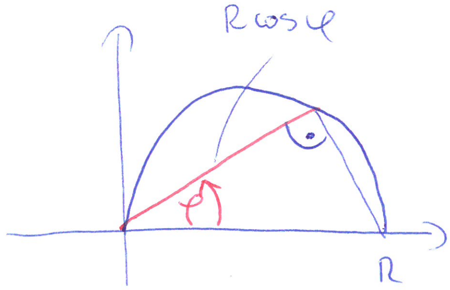
$z=\sqrt{R^2-x^2-y^2}$

felső gömbhéj

$$\frac{m_J(V)}{4} = \iint_T \sqrt{R^2-x^2-y^2}\,dx\,dy \underset{\text{polárkoord}}{=} \int_0^{\pi/2}\int_0^{R\cos\varphi}\sqrt{R^2-r^2}\,\underbrace{r}_{\text{Jacobi-det}}\,dr\,d\varphi =$$

$x = r\cos\varphi$ $\quad$ $T:\ 0\le r\le R\cos\varphi$

$y = r\sin\varphi$ $\quad$ $0\le\varphi\le\frac{\pi}{2}$



$$= -\frac{1}{2}\int_0^{\pi/2}\underbrace{\int_0^{R\cos\varphi}(-2r)(R^2-r^2)^{1/2}\,dr}\,d\varphi \;(=)$$

$$\left[\frac{(R^2-r^2)^{3/2}}{3/2}\right]_{r=0}^{R\cos\varphi} =$$

$$= \frac{2}{3}\left(\big(\underbrace{R^2-R^2\cos^2\varphi}_{R^2\sin^2\varphi}\big)^{3/2}-R^3\right) =$$

$$= \frac{2}{3}R^3\left(\sin^3\varphi-1\right)$$

$$(=)\ \frac{R^3}{3}\int_0^{\pi/2}\left(1-\sin^3\varphi\right)d\varphi = \dots = \frac{R^3}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right)$$

$$\Rightarrow \boxed{m_J(V) = \frac{4R^3}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right)}$$

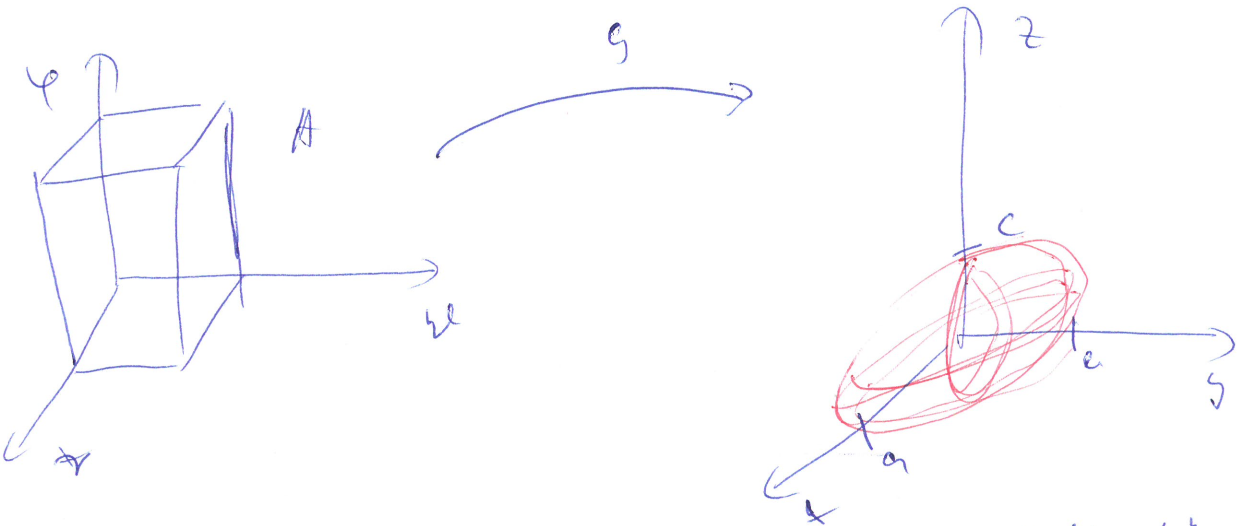
HF: adjuk meg gömbi koordinátákat hengerkoordinátákkal!

Elliptikus koordináták

$a, b, c > 0$

$$\left.\begin{aligned} x &= a\, r \sin\vartheta \cos\varphi \\ y &= b\, r \sin\vartheta \sin\varphi \\ z &= c\, r \cos\vartheta \end{aligned}\right\} \qquad \left.\begin{aligned} 0 &\le r \le 1 \\ 0 &\le \varphi \le 2\pi \\ 0 &\le \vartheta \le \pi \end{aligned}\right\} A$$

$$g(r,\vartheta,\varphi) = \begin{pmatrix} a\, r \sin\vartheta \cos\varphi \\ b\, r \sin\vartheta \sin\varphi \\ c\, r \cos\vartheta \end{pmatrix}$$



a, b, c féltengelyű ellipszoid

$$\det g'(r,\vartheta,\varphi) = \left|\frac{\partial(x,y,z)}{\partial(r,\vartheta,\varphi)}\right| = \underline{\underline{abc\, r^2 \sin\vartheta}}$$

Jacobi-det.

367)

Pl | $V: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$ ellipsoid

$$I := \int_V \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right) dV = ?$$

elliptikus koordinátákban : $V:$ $0\le r\le 1$

$0\le \vartheta\le \pi$

$0\le \varphi\le 2\pi$

$$I = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^2\, abc\, r^2 \sin\vartheta \, dr\, d\vartheta\, d\varphi \;(=)$$

$$\frac{a^2r^2\sin^2\vartheta\cos^2\varphi}{a^2}+\frac{b^2r^2\sin^2\vartheta\sin^2\varphi}{b^2}+\frac{c^2r^2\cos^2\vartheta}{c^2}=$$

$$=\underbrace{r^2\sin^2\vartheta\cos^2\varphi+r^2\sin^2\vartheta\sin^2\varphi}_{r^2\sin^2\vartheta}+r^2\cos^2\vartheta = r^2$$

$$(=)\;\underbrace{\int_0^1 r^4\,dr}_{\left[\frac{r^5}{5}\right]_0^1=\frac{1}{5}}\cdot\underbrace{\int_0^{\pi}\sin\vartheta\, d\vartheta}_{\left[-\cos\vartheta\right]_0^{\pi}=2}\cdot\underbrace{\int_0^{2\pi} d\varphi}_{2\pi}\cdot \cancel{abb}\; abc = \frac{4\pi abc}{5}$$

368/

Példa Határozzuk meg az $(x^2+y^2+z^2)^n = x^{2n-1}$ felülettel határolt test térfogatát!

módosított gömbi koordináták $\rightsquigarrow$ felcseréljük a tengelyeket:

$$z \to x = r\cos\vartheta$$
$$x \to y = r\sin\vartheta\cos\varphi$$
$$y \to z = r\sin\vartheta\sin\varphi$$

↳ Jacobi-det nem változik $\left|\dfrac{\partial(x,y,z)}{\partial(r,\vartheta,\varphi)}\right| = r^2\sin\vartheta$

a felület egyenlete ezekben a koordinátákban:

$$r^{2n} = r^{2n-1}\cdot\cos^{2n-1}\vartheta \implies \text{a határon:}$$

$$\boxed{r = \cos^{2n-1}\vartheta}$$

$r>0 \rightsquigarrow \quad 0\le\vartheta\le\frac{\pi}{2}$

$\Downarrow$

$$m_J(V) = \int_0^{2\pi}\int_0^{\pi/2}\underbrace{\int_0^{\cos^{2n-1}\vartheta} r^2\sin\vartheta\,dr}_{\left[\frac{r^3}{3}\sin\vartheta\right]_0^{\cos^{2n-1}\vartheta} = \frac{1}{3}\cos^{6n-3}\vartheta\sin\vartheta}\,d\vartheta\,d\varphi \;=$$

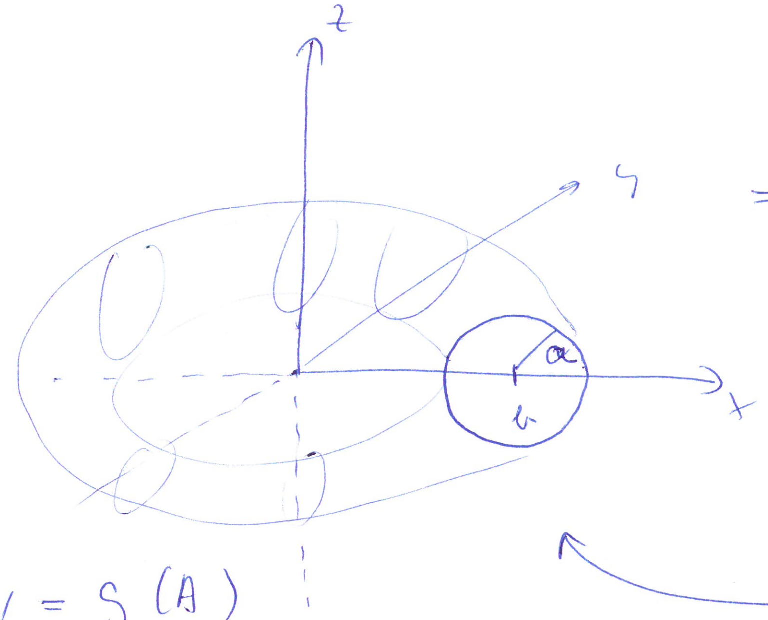
$$= \frac{2\pi}{3}\int_0^{\pi/2}\cos^{6n-3}\vartheta\sin\vartheta\,d\vartheta = -\frac{2\pi}{3}\underbrace{\left[\frac{\cos^{6n-2}\vartheta}{6n-2}\right]_0^{\pi/2}}_{\frac{1}{6n-2}(0-1)} = \underline{\underline{\frac{\pi}{3(3n-1)}}}.$$

365)

Példa **Tórusz térfogata**

$$g(r,\vartheta,\varphi) = \big(\underbrace{(b+r\cos\varphi)\cdot\cos\vartheta}_{x},\ \underbrace{(b+r\cos\varphi)\sin\vartheta}_{y},\ \underbrace{r\sin\varphi}_{z}\big)$$

az (x,z) síkon a $(b,0)$ középpontú a sugarú kört megforgatjuk a z tengely körül



$$\Longrightarrow \left.\begin{array}{l} 0 \le r \le a \\ 0 \le \vartheta \le 2\pi \\ 0 \le \varphi \le 2\pi \end{array}\right\} A$$

g

$V = g(A)$

Jacobi-det. $\left| \dfrac{\partial(x,y,z)}{\partial(r,\vartheta,\varphi)} \right| = r(b + r\cos\varphi)$

$$m_J(V) = \int_0^a \int_0^{2\pi} \underbrace{\int_0^{2\pi} r(b+r\cos\varphi)\,d\varphi}_{\left[rb\varphi + r^2\sin\varphi\right]_0^{2\pi} = 2\pi r b}\, d\vartheta\, dr = 2\pi\cdot 2\pi b \underbrace{\int_0^a r\,dr}_{\left[\frac{r^2}{2}\right]_0^a = \frac{a^2}{2}} \ominus$$

$$\ominus\ \underline{2\pi^2 a^2 b}$$

Többes integrálok néhány alkalmazása

$V \subset \mathbb{R}^3$, $\rho(x,y,z)$ sűrűségű V test tömege:

$$M = \iiint_V \rho(x,y,z)\, dV$$

- tömegközéppontjának koordinátái:

$$x_0 = \frac{\int_V x \rho\, dV}{M}, \quad y_0 = \frac{\int_V y \rho\, dV}{M}, \quad z_0 = \frac{\int_V z \rho\, dV}{M}$$

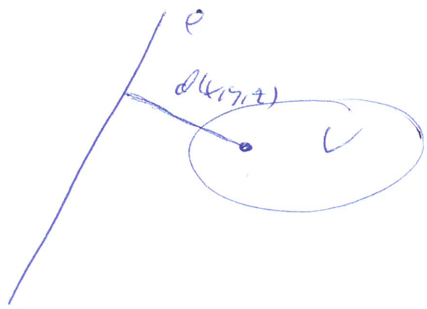
megj: 1) $T \subset \mathbb{R}^2$ esetén $\rightsquigarrow$ $x_0 = \dots$, $y_0 = \dots$
(lemez)

2) $\rho = c$ állandó $\longrightarrow$ tömegközéppont = súlypont.

- $V \subset \mathbb{R}^3$ $e \subset \mathbb{R}^3$ egyenesre vonatkoztatott másodrendű nyomatéka:

$$\int_V \rho(x,y,z)\, d^2(x,y,z)\, dV \quad \text{, ahol}$$

$d(x,y,z)$ az (x,y,z) pont e egyenestől vett távolsága

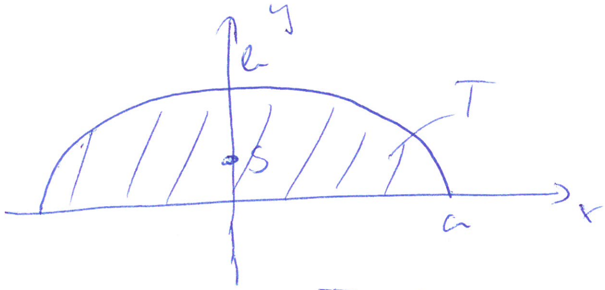


371)

Spec. $e = z$ tengelyre vett nyomaték:

$$\int_V (x^2+y^2)\,\varrho(x,y,z)\,dV$$

Példa $T:\ \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\le 1,\ y\ge 0$ félellipszis súlypontja?



$x_0 = 0$ (szimmetria miatt)

$$M=\int_T dT = \int_P = \int_0^{\pi}\int_0^1 abr\,dr\,d\varphi = \pi ab \underbrace{\int_0^1 r\,dr}_{\left[\frac{r^2}{2}\right]_0^1=\frac{1}{2}} = \frac{\pi ab}{2}$$

$$\begin{cases} x = ar\cos\varphi \\ y = br\sin\varphi \end{cases} \qquad \left|\frac{\partial(x,y)}{\partial(r,\varphi)}\right| = abr$$

$0\le r\le 1$

$0\le\varphi\le\pi$

$$\int_T y\,dT = \int_0^{\pi}\int_0^1 br\sin\varphi\cdot abr\,dr\,d\varphi = ab^2\underbrace{\int_0^1 r^2\,dr}_{\left[\frac{r^3}{3}\right]_0^1=\frac{1}{3}}\cdot\underbrace{\int_0^{\pi}\sin\varphi\,d\varphi}_{\left[-\cos\varphi\right]_0^{\pi}=2} =$$

$$=\frac{2ab^2}{3}$$

$$\Rightarrow\quad y_0 = \frac{\frac{2ab^2}{3}}{\frac{\pi ab}{2}} = \frac{2ab^2}{3}\cdot\frac{2}{\pi ab} = \frac{4}{3}\frac{b}{\pi}$$

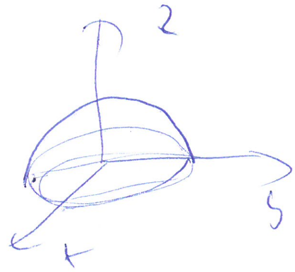
súlypont:

$$S\left(0, \frac{4}{3}\frac{b}{\pi}\right)$$

372)

Példa $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}\le 1$ $z\ge 0$ homogén félellipszoid

súlypontja?



szimmetria $\rightsquigarrow$ $x_0=0$, $y_0=0$

$$\left.\begin{aligned} x&=a r\sin\vartheta\cos\varphi\\ y&=b r\sin\vartheta\sin\varphi\\ z&=c r\cos\vartheta\end{aligned}\right\}\qquad \left|\frac{\partial(x,y,z)}{\partial(r,\vartheta,\varphi)}\right|=abc\,r^2\sin\vartheta$$

$$0\le r\le 1\ ,\ 0\le\vartheta\le\frac{\pi}{2},\ 0\le\varphi\le 2\pi$$

$$M=\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 abc\,r^2\sin\vartheta\,dr\,d\vartheta\,d\varphi=\dots=\frac{2abc\pi}{3}$$

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \underbrace{cr\cos\vartheta}_{z}\,abc\,r^2\sin\vartheta\,dr\,d\vartheta\,d\varphi=\frac{abc^2\pi}{4}$$

$$\hookrightarrow\quad z_0=\frac{abc^2\pi}{4}\cdot\frac{3}{2abc\pi}=\underline{\underline{\frac{3}{4}c}}$$