1)

A határozatlan integrál I.

Def: Ha F diffható I intervallumon és $F'(x) = f(x)$ $\forall x \in I$-re, akkor F az f primitív függvénye.

TÉTEL: Ha F primitív függvénye f-nek, akkor f összes primitív fv-e $F+c$ alakú, c konstans.

$\Downarrow$

Def: Az f fv primitív fv-einek összességét $\int f\,dx$-nel jelöljük és f határozatlan integráljának hívjuk.

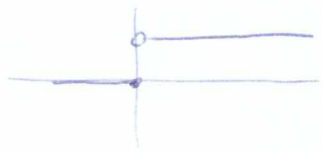
$$\hookrightarrow \boxed{\int f\,dx = F(x) + c}$$

$$F'(x) = \frac{dF}{dx} = f$$

Nem minden fv-nek $\exists$ primitív függvénye

Pl:

$$f(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases}$$



tfh $F(x)$ primitív fv $\Rightarrow F'(x) = 0$, ha $x < 0$

$$\Downarrow$$

$$F(x) = c, \text{ ha } x \le 0$$

$\Rightarrow F'(x) = 1$, ha $x > 0$

$$\Downarrow$$

$$F(x) = x + c \quad x \ge 0$$

$\Rightarrow F'_-(0) = 0$, $F'_+(0) = 1$ $\Rightarrow F$ nem diffható 0-ban

Megj.: Ha f folytonos I intervallumon $\Rightarrow \exists$ primitív függvénye (Később) (nem szükséges feltétel!)

Alapintegrálok

$$\int x^\alpha \, dx = \frac{x^{\alpha+1}}{\alpha+1} + C \quad (\alpha \neq -1)$$

$$\int \frac{1}{x} \, dx = \ln|x| + C$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + C \quad (a \neq 1)$$

$$\int e^x \, dx = e^x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \frac{1}{\cos^2 x} \, dx = \operatorname{tg} x + C$$

$$\int \frac{1}{\sin^2 x} \, dx = -\operatorname{ctg} x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C$$

$$\int \frac{1}{1+x^2} \, dx = \operatorname{arctg} x + C$$

$$\int \operatorname{ch} x \, dx = \operatorname{sh} x + C$$

$$\int \operatorname{sh} x \, dx = \operatorname{ch} x + C$$

$$\int \frac{1}{\operatorname{ch}^2 x} \, dx = \operatorname{th} x + C$$

$$\int \frac{1}{\operatorname{sh}^2 x} \, dx = -\operatorname{cth} x + C$$

$$\int \frac{1}{\sqrt{x^2+1}} \, dx = \operatorname{arsh} x + C$$

$$\int \frac{1}{\sqrt{x^2-1}} \, dx = \operatorname{arch} x + C$$

$$\int \frac{1}{1-x^2} \, dx = \frac{1}{2} \ln\left|\frac{1+x}{1-x}\right| + C$$

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TÉTEL Ha f-nek és g-nek $\exists$ primitív fv I-n $\Rightarrow$ $f+g$-nek és $c\cdot f$-nek is $\exists$ és

$$i)\ \int (f+g)\,dx = \int f\,dx + \int g\,dx$$

$$ii)\ \int c f\,dx = c\int f\,dx$$

Biz:
$$\left.\begin{array}{l}\int f(x)\,dx = F + c\\ \int g(x)\,dx = G + c\end{array}\right\} \Rightarrow \begin{array}{l}(F+G)' = f+g \Rightarrow i) \checkmark\\ (cF)' = c\cdot F' \Rightarrow ii) \checkmark\end{array}$$

Példák:

① $$\int (x^3+2x-3)\,dx = \frac{x^4}{4} + x^2 - 3x + C$$

② $$\int \frac{(x+1)^2}{x^3}\,dx = \int \frac{x^2+2x+1}{x^3}\,dx = \int \left(\frac{1}{x} + 2\frac{1}{x^2} + \frac{1}{x^3}\right)dx =$$

$$= \ln|x| + 2\cdot\left(-\frac{1}{x}\right) - \frac{1}{2x^2} + c$$

TÉTEL:
$$\int f(x)\,dx = F + c \Rightarrow \boxed{\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + c}$$

Biz:
$$\left[\frac{1}{a}F(ax+b)+c\right]' = \frac{1}{a}F'(ax+b)\cdot a = f(ax+b) \quad \circ!$$

Példák.

① $$\int \sqrt{2x-3}\,dx = \int (2x-3)^{1/2}\,dx = \frac{1}{2}\,\frac{(2x-3)^{3/2}}{3/2} + c$$

② $$\int e^{-x}\,dx = \frac{e^{-x}}{-1} + c = -e^{-x} + c$$

③ $$\int \sin\frac{x+1}{3}\,dx = -3\cos\frac{x+1}{3} + c$$

④ $$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \int\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)dx =$$

$$= \frac{1}{2}x - \frac{1}{2}\cdot\frac{\sin 2x}{2} + c$$

⑤ $$\int \sin^3 x\,dx = \int \sin^2 x\cdot\sin x\,dx = \int \frac{1-\cos 2x}{2}\sin x\,dx =$$

$$= \int\left(\frac{1}{2}\sin x - \frac{1}{2}\cos 2x\sin x\right)dx \quad =$$

<u>eml</u>:

$$\sin x\cos y = \frac{1}{2}\left(\sin(x+y) + \sin(x-y)\right)$$

$$\sin x\sin y = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$$

$$\cos x\cos y = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right)$$

$$= \int\left(\frac{1}{2}\sin x - \frac{1}{4}\left(\sin 3x + \sin(-x)\right)\right)dx =$$

$$= \int\left(\frac{3}{4}\sin x - \frac{1}{4}\sin 3x\right)dx = -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + C$$

TÉTEL: Ha f deriválható és mindenütt pozitív I-n, akkor

$$\boxed{\int [f(x)]^\alpha f'(x)\,dx = \frac{[f(x)]^{\alpha+1}}{\alpha+1} + C \qquad \alpha \neq -1}$$

$$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C}$$

Biz: házilag

Példák:

(1) $\int \cos x \sin^3 x\,dx = \frac{\sin^4 x}{4} + C$

(2) $\int \operatorname{tg} x\,dx = -\int \frac{-\sin x}{\cos x}\,dx = -\ln|\cos x| + C$

(3) $\int x\sqrt{1+x^2}\,dx = \int \frac{1}{2}\sqrt{1+x^2}\cdot 2x\,dx =$

$= \frac{1}{2}\int (x^2+1)^{1/2}(x^2+1)'\,dx = \frac{1}{2}\cdot\frac{2}{3}(x^2+1)^{3/2} + C$

(4) $\int \frac{x}{1+x^2} = \frac{1}{2}\int \frac{(x^2+1)'}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1) + C$

TÉTEL (parciális integrálás)

Tfh f és g diff.hatóak I-n és itt $\exists$ $f\cdot g'$-nek a primitív fv-e. Ekkor $f'g$-nek is $\exists$ primitív fv-e és

$$\boxed{\int f'g\,dx = f\cdot g - \int f g'\,dx}$$

Biz.:

$$\int f\cdot g' = F + c$$

$$(f\cdot g)' = f'\cdot g + f\cdot g' \Rightarrow (fg - F)' = f'g + fg' - fg' = f'g$$

Példák: (1) $\{\text{polinom}\}\cdot\{\cos, \sin, \exp\}$

- $\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$

$f' = \cos x \quad g = x$

$f = \sin x \quad g' = 1$

(2) $\int x e^x\,dx = x e^x - \int e^x\,dx = x e^x - e^x + C$

$f' = e^x \quad g = x$

$f = e^x \quad g' = 1$

4/ ② $\{\cos x, \sin x, \exp\} \times \{\cos, \sin, \exp\}$

$$\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$$

$f' = e^x \quad g = \cos x \qquad\qquad f' = e^x \quad g = \sin x$

$f = e^x \quad g' = -\sin x \qquad\qquad f = e^x \quad g' = \cos x$

$$\Rightarrow \int e^x \cos x \, dx = \frac{1}{2}(e^x \cos x + e^x \sin x) + C$$

③ $$\int \ln x \, dx = \int 1 \cdot \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - x + C$$

$f' = 1 \quad g = \ln x$

$f = x \quad g' = \frac{1}{x}$

④ $$\int \operatorname{arctg} x \, dx = \int 1 \cdot \operatorname{arctg} x \, dx = x \operatorname{arctg} x - \underbrace{\int \frac{x}{1+x^2} \, dx}_{\frac{1}{2} \underbrace{\int \frac{2x}{1+x^2} \, dx}_{\ln(1+x^2) + C}}$$

$f' = 1 \quad g = \operatorname{arctg} x$

$f = x \quad g' = \frac{1}{1+x^2}$

⑤ $$\int x \cdot \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x}{2} \, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$$

$f' = x \quad g = \ln x$

$f = \frac{x^2}{2} \quad g' = \frac{1}{x}$

Helyettesítéses integrálás

TÉTEL: Tfh g diffható I-n és f értelmezve van $\mathcal{J} = g(I)$-n, f-nek F primitív f-e $\mathcal{J}$-n. Ekkor $(f \circ g)\cdot g'$-nek is F primitív f-e I-n és

$$\boxed{\int f(g(x))\cdot g'(x)\,dx = F(g(x)) + c \qquad \text{ahol} \quad \int f\,dx = F(x) + c}$$

Biz:

$$\left[F(g(x)) + c\right]' = F'(g(x))\cdot g'(x) = f(g(x))\cdot g'(x) \qquad \circ!$$

Példák

(1)

$$\int t\, e^{t^2} dt = \frac{1}{2}\int e^{t^2}\cdot 2t\, dt = \frac{1}{2}\int e^{x} dx = \frac{1}{2} e^{x} + c = \frac{1}{2} e^{t^2} + c$$

$x = t^2$

$\frac{dx}{dt} = 2t$

$2t\,dt = dx$

megj: általában:

$$\boxed{\int e^{f(x)}\cdot f'(x)\,dx = e^{f(x)} + c}$$

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(2)

$$\int \sqrt{x^2+6x+18}\, dx = \int \sqrt{(x+3)^2+9}\, dx = 3\int \sqrt{\left(\frac{x+3}{3}\right)^2+1}\, dx \Leftrightarrow$$

$$y = \frac{x+3}{3} \qquad \frac{dy}{dx} = \frac{1}{3}$$

$$\Leftrightarrow 9\int \sqrt{y^2+1}\, dy$$

$y = \operatorname{sh} t$ ....

R: racionális törtfv-ek

Áll | $\boxed{\int R(e^x)\, dx \text{ alakú integrálok } t=e^x \text{ helyettesítéssel rac. törtfv-el}}$

$$t = e^x \to x = \ln t \to \frac{dx}{dt} = \frac{1}{t}$$

Pl |

$$\int \frac{4}{e^{2x}-4}\, dx = \int \frac{4}{t^2-4}\,\frac{1}{t}\, dt = \int \frac{4}{t(t+2)(t-2)}\, dt \Leftrightarrow$$

$$\frac{4}{t(t+2)(t-2)} = \frac{A}{t} + \frac{B}{t+2} + \frac{C}{t-2}$$

$$A = -1,\ B = \frac{1}{2},\ C = \frac{1}{2}$$

$$\Leftrightarrow \int \left(-\frac{1}{t} + \frac{1}{2}\frac{1}{t+2} + \frac{1}{2}\frac{1}{t-2}\right) dt =$$

$$= -\ln|t| + \frac{1}{2}\ln(t+2) + \frac{1}{2}\ln|t-2| + c =$$

$$= -\ln e^x + \frac{1}{2}\ln(e^x+2) + \frac{1}{2}\ln|e^x-2| + c$$

Trigonometrikus fv-ek integrálása

Áll. R(sin x, cos x) alakú fv-ek $\boxed{t = \operatorname{tg}\frac{x}{2}}$ helyettesítéssel rac. törtfv-ek.

$$\sin^2 x + \cos^2 x = 1 \longrightarrow \operatorname{tg}^2 x + 1 = \frac{1}{\cos^2 x}$$

$$\Downarrow$$

$$\cos x = \frac{1}{\pm\sqrt{1+\operatorname{tg}^2 x}} \qquad \sin x = \frac{\operatorname{tg} x}{\pm\sqrt{1+\operatorname{tg}^2 x}}$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}}$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1-\operatorname{tg}^2\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}}$$

$$t = \operatorname{tg}\frac{x}{2} \implies \boxed{\sin x = \frac{2t}{1+t^2}} \qquad \boxed{\cos x = \frac{1-t^2}{1+t^2}}$$

$$x = 2\operatorname{arctg} t \rightarrow \boxed{\frac{dx}{dt} = \frac{2}{1+t^2}}$$

Példák.

(1) $$\int \frac{dx}{\sin x} = \int \frac{1+t^2}{2t}\,\frac{2}{1+t^2}\,dt = \int \frac{dt}{t} = \log|t| + c =$$

$$= \log\left|\operatorname{tg}\frac{x}{2}\right| + c$$

Hasonlóan:

$$\int \frac{dx}{\cos x} = \int \frac{dx}{\sin\left(\frac{\pi}{2} - x\right)} = -\ln\left|\operatorname{tg}\left(\frac{\pi}{4} - \frac{x}{2}\right)\right| + c$$

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trigonometrikus fv-ek páros hatványa:

$$\int \cos^4 x \, dx = \int (\cos^2 x)^2 dx = \int \left(\frac{1+\cos 2x}{2} \right)^2 dx =$$

$$= \frac{1}{4} \int \left(1 + 2\cos 2x + \cos^2 2x \right) dx =$$

$$= \frac{1}{4} x + \frac{1}{4} \sin 2x + \underbrace{\frac{1}{4} \int \cos^2 2x \, dx}_{\int \frac{1+\cos 4x}{2} dx = \frac{x}{2} + \frac{\sin 4x}{8} + c} \; \textcircled{=}$$

$$\textcircled{=} \; \frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C$$

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$$\int \frac{1}{\sin^6 x} dx = \int \frac{1}{\sin^4 x} \cdot \frac{1}{\sin^2 x} dx \underset{\uparrow}{=} \int (1+\operatorname{ctg}^2 x)^2 \cdot \frac{1}{\sin^2 x} dx =$$

$$1 + \operatorname{ctg}^2 x = 1 + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$

$$\underset{\uparrow}{=} - \int (1+y^2)^2 dy = - \int (1 + 2y^2 + y^4) dy =$$

$y = \operatorname{ctg} x$

$$= - \left(y + \frac{2}{3} y^3 + \frac{y^5}{5} \right) + C =$$

$$\frac{dy}{dx} = - \frac{1}{\sin^2 x}$$

$$= - \left(\operatorname{ctg} x + \frac{2}{3} \operatorname{ctg}^3 x + \frac{\operatorname{ctg}^5 x}{5} \right) + C$$

1/

A határozatlan integrál II.

1. Racionális törtfüggvények integrálása

<u>Def</u>: *elemi törtek*:

i) $\dfrac{A}{(x-a)^n}$ $\quad n \in \mathbb{N}^+,\ A, a \in \mathbb{R}$

ii) $\dfrac{Ax+B}{(x^2+ax+b)^n}$ $\quad n \in \mathbb{N}^+ \quad A, B, a, b \in \mathbb{R}$

$a^2 - 4b < 0$

(x^2+ax+b-nek $\nexists$ valós gyöke)

i) • $\displaystyle\int \frac{A}{(x-a)}\,dx = A \cdot \ln|x-a| + c$

• $\displaystyle\int \frac{A}{(x-a)^n}\,dx = A\,\frac{(x-a)^{-n+1}}{-n+1} + c = \frac{A}{1-n} \cdot \frac{1}{(x-a)^{n-1}} + c$

ii) $$\int \frac{Ax+B}{(x^2+ax+b)^n}\,dx = \int \frac{(2x+a)\cdot\frac{A}{2} + \left(B - \frac{aA}{2}\right)}{(x^2+ax+b)^n}\,dx =$$

$$= \frac{A}{2}\int \frac{(x^2+ax+b)'}{(x^2+ax+b)^n}\,dx + \left(B - \frac{aA}{2}\right)\int \frac{dx}{(x^2+ax+b)^n}\,dx$$

$\cancel{\frac{A}{2} \ln(x^2+ax+b) + c}$

• $n=1$ $\quad \displaystyle\frac{A}{2}\int \frac{(x^2+ax+b)'}{x^2+ax+b}\,dx = \frac{A}{2}\ln(x^2+ax+b) + c$

• $n>1$ $\quad \displaystyle\frac{A}{2}\int \frac{(x^2+ax+b)'}{(x^2+ax+b)^n}\,dx = \frac{1}{1-n}\,\frac{1}{(x^2+ax+b)^{n-1}} + c$

$$\int \frac{dx}{(x^2+ax+b)^n} = \int \frac{dx}{\left[\left(x+\left(\frac{a}{2}\right)\right)^2 + b - \frac{a^2}{4}\right]^n} =$$

$$= \int \frac{dx}{\left[\left(x+\frac{a}{2}\right)^2 + d^2\right]^n}$$

$$d = \sqrt{b - \frac{a^2}{4}}$$

ha: $\boxed{\int \frac{dx}{(x^2+1)^n} = F_n(x) + c}$

$\hookrightarrow$ $$\int \frac{dx}{\left[\left(x+\frac{a}{2}\right)^2 + d^2\right]^n} = \frac{1}{d} \int \frac{dx}{\left[\left(\frac{x}{d} + \frac{a}{2d}\right)^2 + 1\right]^n} = F_n\left(\frac{x}{d} + \frac{a}{2d}\right) + c$$

áll: $F_1(x) = \operatorname{arctg} x + c$

• $n \geq 1$: $\boxed{F_{n+1} = \frac{1}{2n} \frac{x}{(x^2+1)^n} + \frac{2n-1}{2n} \cdot F_n + c}$

deriválással:

$$F'_{n+1} = \frac{1}{2n} \frac{(x^2+1)^n - x \cdot n (x^2+1)^{n-1} \cdot 2x}{(x^2+1)^{2n}} + \frac{2n-1}{2n} \cdot F'_n =$$

$$= \frac{1}{2n} \frac{x^2+1-2nx^2}{(x^2+1)^{n+1}} + \frac{2n-1}{2n} F'_n = \frac{1}{2n} \left\{ \frac{1+x^2(1-2n)}{(x^2+1)^{n+1}} + (2n-1) F'_n \right\}$$

2)

$$2n\int \frac{x^2}{(x^2+1)^{n+1}}\,dx = -\int x\underbrace{\left(\frac{1}{(x^2+1)^n}\right)'}_{\frac{-n}{(x^2+1)^{n+1}}\cdot 2x}\,dx =$$

$$= -\left[\frac{x}{(x^2+1)^n}\right] + \underbrace{\int \frac{dx}{(x^2+1)^n}}_{F_n(x)} + C$$

$$F_{n+1}' = \frac{1}{(x^2+1)^{n+1}} \overset{?}{=} \frac{1}{2n}\,\frac{x^2(1-2n)+1}{(x^2+1)^{n+1}} + \frac{2n-1}{2n}\,\overbrace{\frac{1}{(x^2+1)^n}}^{F_n'} =$$

$$= \frac{1}{2n}\cdot\frac{1}{(x^2+1)^n}\left\{\underbrace{\frac{x^2(1-2n)+1}{x^2+1} + 2n-1}_{\frac{\cancel{x^2}-\cancel{2nx^2}+\cancel{1}+\cancel{2nx^2}-\cancel{x^2}+2n-\cancel{1}}{x^2+1} = \frac{2n}{x^2+1}}\right\} =$$

$$= \frac{1}{(x^2+1)^{n+1}} \quad .!$$

Példa:

$$F_1(x) = \int \frac{dx}{(x^2+1)} = \operatorname{arctg} x + C$$

$$F_2(x) = \int \frac{dx}{(x^2+1)^2} = \frac{1}{2}\,\frac{x}{x^2+1} + \frac{1}{2}\operatorname{arctg} x + C$$

$$F_3(x) = \int \frac{dx}{(x^2+1)^3} = \frac{1}{4}\,\frac{x}{(x^2+1)^2} + \frac{3}{8}\,\frac{x}{x^2+1} + \frac{3}{8}\operatorname{arctg} x + C$$

$$\vdots$$

TÉTEL: (Parciális törtekre bontás tétele)

$\forall$ R racionális törtfv. előáll egy polinom és véges sok elemi tört összegeként úgy, hogy a szereplő elemi törtek nevezői mind osztói R nevezőjének. Azaz

$$R(x) = \frac{P(x)}{Q(x)} = E(x) + \sum_{i=1}^{\alpha_1} \frac{A_{1i}}{(x-x_1)^i} + \dots + \sum_{i=1}^{\alpha_r} \frac{A_{ri}}{(x-x_r)^i} + \sum_{k=1}^{\beta_1} \frac{B_{1k}x + C_{1k}}{(x^2+b_1x+c_1)^k} + \dots + \sum_{k=1}^{\beta_s} \frac{B_{sk}x + C_{sk}}{(x^2+b_sx+c_s)^k}$$

P, Q polinomok, $E(x)$ polinom

ahol $Q(x) = (x-x_1)^{\alpha_1} \dots (x-x_r)^{\alpha_r} (x^2+b_1x+c_1)^{\beta_1} \dots (x^2+b_sx+c_s)^{\beta_s}$

ahol A_{jk}, B_{jk}, C_{jk} konstansok.

Nem bizonyítjuk.

Példa: (1) $\displaystyle\int \frac{x+2}{x(x^2+1)^2}\,dx = ?$

$$\frac{x+2}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+1)^2} = \frac{A(x^2+1)^2+(Bx+C)x(x^2+1)+(Dx+E)x}{x(x^2+1)^2} =$$

$$= \frac{A(x^4+2x^2+1)+(Bx+C)(x^3+x)+(Dx^2+Ex)}{x(x^2+1)^2}$$

$$\Rightarrow \left.\begin{array}{ll} x^4: & A+B=0 \\ x^3: & C=0 \\ x^2: & 2A+B+D=0 \\ x: & C+E=1 \\ \text{const}: & A=2 \end{array}\right\} \quad A=2,\ B=-2,\ C=0,\ D=-2,\ E=1$$

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$$\int \frac{x+2}{x(x^2+1)^2}\,dx = \int \frac{2}{x}\,dx + \int \underset{\uparrow \frac{f'}{f}}{\frac{-2x}{x^2+1}}\,dx + \int \frac{-2x+1}{(x^2+1)^2}\,dx =$$

$$= 2\ln|x| - \ln(x^2+1) - \int \underset{\overset{\shortparallel}{f'\cdot f^{-2}}}{\frac{2x}{(x^2+1)^2}}\,dx + \int \frac{1}{(x^2+1)^2}\,dx$$

$$-\frac{1}{x^2+1} + \frac{1}{2}\,\frac{x}{x^2+1} + \frac{1}{2}\,\mathrm{arctg}\,x + C$$

(2)

$$\int \frac{2x^2}{x^4-1}\,dx = \int \frac{2x^2}{(x^2-1)(x^2+1)}\,dx = \int \frac{2x^2}{(x-1)(x+1)\,(x^2+1)}\,dx \ominus$$

$$\frac{2x^2}{(x-1)(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}$$

$$A = -\frac{1}{2},\ B = \frac{1}{2},\ C = 0,\ D = 1$$

$$\ominus -\frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{1}{2}\int \frac{1}{x-1}\,dx + \int \frac{1}{x^2+1}\,dx =$$

$$= -\frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + \mathrm{arctg}\,x + C$$

Gyökös kifejezéseket tartalmazó integrálok

$$R(u,v) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{n} a_{ij} u^i v^j}{\sum_{i=0}^{n}\sum_{j=0}^{n} b_{ij} u^i v^j} \qquad n \geq 0,\ n \in \mathbb{Z}$$

$\uparrow$ rac. törtfv

① Áll.: $\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right)$ $\quad (c^2+d^2 \neq 0,\ ad \neq bc)$

$t = \sqrt[n]{\frac{ax+b}{cx+d}}$ helyettesítéssel racionális törtfv integráljára vezet.

Biz.:

$$\frac{ax+b}{cx+d} = t^n \rightarrow ax+b = ct^n x + dt^n$$

$$x = \frac{dt^n - b}{a - ct^n} \quad \text{rac. törtfv}$$

$\frac{dx}{dt}$ is rac. törtfv. $\circ$!

Pl. ①

$$\int \frac{1}{x^2}\sqrt[3]{\frac{x+1}{x}}\, dx \ (=)$$

$$t := \sqrt[3]{\frac{x+1}{x}} \rightarrow x+1 = t^3 x \rightarrow x = \frac{1}{t^3-1}$$

$$\Downarrow$$

$$\frac{dx}{dt} = -\frac{3t^2}{(t^3-1)^2}$$

$$(=) \int (t^3-1)^2 \cdot t \, \frac{-3t^2}{(t^3-1)^2}\, dt = \int -3t^3\, dt = -\frac{3}{4}t^4 + C =$$

$$= -\frac{3}{4}\left(\frac{x+1}{x}\right)^{4/3} + C$$

4/

② $$\int \sqrt{\frac{x-3}{x-1}}\,dx \;\textcircled{=}$$

$$t := \sqrt{\frac{x-3}{x-1}} \Rightarrow t^2(x-1) = x-3 \Rightarrow x = \frac{t^2-3}{t^2-1}$$

$$\frac{dx}{dt} = \frac{2t(t^2-1)-2t(t^2-3)}{(t^2-1)^2} = \frac{4t}{(t^2-1)^2}$$

$$\textcircled{=} \int \frac{4t^2}{(t^2-1)^2}\,dt \underset{\uparrow}{=} \int \left(\frac{1}{(t+1)^2} - \frac{1}{t+1} + \frac{1}{(t-1)^2} + \frac{1}{t-1} \right) dt \;\boxed{=}$$

$$\frac{4t^2}{(t-1)^2(t+1)^2} = \frac{A}{(t+1)^2} + \frac{B}{t+1} + \frac{C}{(t-1)^2} + \frac{D}{t-1}$$

$$A = 1,\ B = -1,\ C = 1,\ D = 1$$

$$\boxed{=} -\frac{1}{t+1} - \ln|t+1| - \frac{1}{t-1} + \ln|t-1| + c =$$

$$= -\frac{2t}{t^2-1} + \ln\left|\frac{t-1}{t+1}\right| + c = \frac{-2\sqrt{\frac{x-3}{x-1}}}{\frac{x-3}{x-1}-1} + \ln\left|\frac{\sqrt{\frac{x-3}{x-1}}-1}{\sqrt{\frac{x-3}{x-1}}+1}\right|$$

② **Áll** $\int R(x, \sqrt[n]{ax+b})\,dx$ integrálok $t = \sqrt[n]{ax+b}$ helyettesítéssel rac. törtfv-nyé alakíthatók

Pl

$$\int x\sqrt{5x+3}\,dx \underset{\uparrow}{=} \int \frac{t^2-3}{5}\cdot t\,\frac{2}{5}t\,dt = \frac{2}{25}\int (t^4-3t^2)\,dt =$$

$$t = \sqrt{5x+3}$$

$$= \frac{2}{25}\left(\frac{t^5}{5} - t^3\right) + c =$$

$$t^2 = 5x+3$$

$$\boxed{x = \frac{t^2-3}{5}} \qquad \boxed{\frac{dx}{dt} = \frac{2}{5}t}$$

$$= \frac{2}{25}\left(\frac{\sqrt{(5x+3)^5}}{5} - \sqrt{(5x+3)^3}\right) + c$$

5)

(2)

$$\int \frac{dx}{\sqrt{x}+1} \underset{\substack{\uparrow \\ x=t^2 \\ \frac{dx}{dt}=2t \\ dx = 2t\,dt}}{=} \int \frac{2t\,dt}{t+1} = 2\int \frac{t+1-1}{t+1}\,dt =$$

$$= 2\int \left(1 - \frac{1}{t+1}\right)dt =$$

$$= 2t - 2\ln|1+t| + C =$$

$$\underset{\substack{\uparrow \\ t=\sqrt{x}}}{=} 2\sqrt{x} - 2\ln(1+\sqrt{x}) + C$$

(3)

$$\int \sqrt{1-x^2}\,dx \underset{\substack{\uparrow \\ x=\sin t\ ,\ t\in[-\pi/2,\pi/2] \\ \frac{dx}{dt}=\cos t \\ dx=\cos t\,dt}}{=} \int \sqrt{1-\sin^2 t}\ \cos t\,dt = \int \cos^2 t\,dt \ (=)$$

$$(=) \int \frac{1+\cos 2t}{2}\,dt = \frac{t}{2} + \frac{\sin 2t}{4} + C \underset{\substack{\uparrow \\ t=\arcsin x}}{=} \frac{\arcsin x}{2} + \frac{\sin(2\arcsin x)}{4} + C$$

$*$(1)

$$\sin(2\arcsin x) = 2\sin(\arcsin x)\cos(\arcsin x) = 2x\sqrt{1-x^2}$$

$$\overset{*}{(=)}\ \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + C$$

(5)

$$\int \frac{e^{2x}}{e^x+1}\,dx \underset{\uparrow}{=} \int \frac{t^2}{t+1}\,\frac{1}{t}\,dt = \int \frac{t}{t+1}\,dt = \int \frac{t+1-1}{t+1}\,dt =$$

$t := e^x$

$\frac{dt}{dx} = e^x = t$

$dx = \frac{1}{t}\,dt$

$$= \int \left(1 - \frac{1}{t+1}\right) dt =$$

$$= t - \ln|t+1| + c$$

$$\cancel{= \ln t - \ln}$$

$$\underset{\cancel{x = \ln t}}{=} e^x - \ln(e^x+1) + c$$

(5)

$$\int \cos^5 x\,dx = \int \cos^4 x \cdot \cos x\,dx = \int (1-\sin^2 x)^2 \cos x\,dx \;(=)$$

$y := \sin x$

$dy = \cos x\,dx$

$$(=) \int (1-y^2)^2\,dy = \int (1 - 2y^2 + y^4)\,dy = y - 2\frac{y^3}{3} + \frac{y^5}{5} + c =$$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{\sin^5 x}{5} + c$$

[illegible] [illegible], ko sin v cos [illegible] [illegible]

(5)

$$\int \frac{e^{2x}}{e^x+1}\,dx \underset{\uparrow}{=} \int \frac{t^2}{t+1}\,\frac{1}{t}\,dt = \int \frac{t}{t+1}\,dt = \int \frac{t+1-1}{t+1}\,dt =$$

$t := e^x$

$\frac{dt}{dx} = e^x = t$

$dx = \frac{1}{t}\,dt$

$$= \int \left(1 - \frac{1}{t+1}\right) dt =$$

$$= t - \ln|t+1| + c$$

$$\cancel{= \ln t - \ln}$$

$$\underset{\cancel{x = \ln t}}{\uparrow} = e^x - \ln(e^x+1) + c$$

(5)

$$\int \cos^5 x\,dx = \int \cos^4 x \cdot \cos x\,dx = \int (1-\sin^2 x)^2 \cos x\,dx =$$

$y := \sin x$

$dy = \cos x\,dx$

$$= \int (1-y^2)^2\,dy = \int (1 - 2y^2 + y^4)\,dy = y - 2\frac{y^3}{3} + \frac{y^5}{5} + c =$$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{\sin^5 x}{5} + c$$

mindig működik, ha sin v cos páratlan hatványon

6/ ...

(2)
$$\int \frac{1+\sin x}{1-\cos x}\,dx = \int \underbrace{\frac{1+\frac{2t}{1+t^2}}{1+\frac{t^2-1}{1+t^2}}}_{\frac{1+t^2+2t}{1+t^2+t^2-1}} \frac{2}{1+t^2}\,dt = \int \frac{t^2+2t+1}{2t^2(1+t^2)}\,dt =$$

$$= \int \frac{t^2+2t+1}{t^2(t^2+1)}\,dt = \int \left(\frac{2}{t} + \frac{1}{t^2} - \frac{2t}{t^2+1} \right) dt = 2\ln|t| - \frac{1}{t} - \ln(t^2+1) + c$$

P

$$\frac{t^2+2t+1}{t^2(t^2+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{Ct+D}{t^2+1} \qquad A=2,\ B=1,\ C=-2,\ D=0$$

$$= 2\ln\left|\operatorname{tg}\frac{x}{2}\right| - \frac{1}{\operatorname{tg}\frac{x}{2}} - \ln\left(1+\operatorname{tg}^2\frac{x}{2}\right) + C$$

Megj: Ha $\sin x$ és $\cos x$ páros hatványon van, célszerűbb:

$$\sin^2 x = \frac{\operatorname{tg}^2 x}{1+\operatorname{tg}^2 x} = \frac{1}{1+\operatorname{ctg}^2 x}$$

$$\cos^2 x = \frac{1}{1+\operatorname{tg}^2 x}$$

Pl:
$$\int \frac{1}{2+\sin^2 x}\,dx = \int \frac{1}{2+\frac{1}{1+\operatorname{ctg}^2 x}}\,dx = \int \frac{1+\operatorname{ctg}^2 x}{3+2\operatorname{ctg}^2 x}\,dx =$$

$$y := \operatorname{ctg} x \qquad x = \operatorname{arc\,ctg} y \qquad dx = -\frac{1}{1+y^2}\,dy$$

$$= -\int \frac{1+y^2}{3+2y^2} \cdot \frac{1}{1+y^2}\,dy = -\int \frac{1}{3+2y^2}\,dy = -\frac{1}{3} \int \frac{1}{1+\left(\sqrt{\frac{2}{3}}\,y\right)^2}\,dy =$$

$$= -\frac{1}{3}\sqrt{\frac{3}{2}}\,\operatorname{arctg}\sqrt{\frac{2}{3}}\,y + c = -\frac{1}{\sqrt{6}}\,\operatorname{arctg}\left(\sqrt{\frac{2}{3}}\,\operatorname{ctg} x\right) + c$$

(6)

$$\int \frac{dx}{(x^2+1)^2} \underset{\uparrow}{=} \int \frac{dt}{\cos^2 t} \, \underbrace{\frac{1}{(tg^2 t+1)^2}}_{\frac{1}{\cos^4 t}} = \int \cos^2 t \, dt \ldots$$

$x = tg\, t$

$dx = \frac{1}{\cos^2 t} dt$

(7)

$$\int \sqrt{1+x^2}\, dx \underset{\uparrow}{=} \int \sqrt{1+sh^2 t}\, ch\, t\, dt = \int ch^2 t\, dt = \int \frac{1+ch\, 2t}{2} dt =$$

$x = sh\, t$

$dx = ch\, t\, dt$

$$= \frac{arsh\, x}{2} + \frac{x\sqrt{1+x^2}}{2}.$$

(8)

$$\int tg^6 x\, dx \underset{\uparrow}{=} \int \frac{y^6}{1+y^2} dy = \int \left(y^4 - y^2 + 1 - \frac{1}{y^2+1}\right) dy =$$

$y = tg\, x \quad x = arctg\, y$

$dx = \frac{1}{1+y^2} dy$

$$= \frac{y^5}{5} - \frac{y^3}{3} + y - arctg\, y =$$

$$= \frac{tg^5 x}{5} - \frac{tg^3 x}{3} + tg\, x - x$$

(9)

$$\int \frac{1}{1+\cos x} dx \underset{\uparrow}{=} \int \frac{1}{1 + \frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2} = \int dt = t = tg \frac{x}{2} + c$$

$t = tg \frac{x}{2}$

$\cos x = \frac{1}{\sqrt{1+tg^2 x}}$

$\sin 2x = 2 \sin x \cos x = \frac{2 tg\, x}{1+tg^2 x}$

$$\Rightarrow \boxed{\begin{aligned} \sin x &= \frac{2 tg \frac{x}{2}}{1+tg^2 \frac{x}{2}} \\ \cos x &= \frac{1-tg^2 \frac{x}{2}}{1+tg^2\frac{x}{2}} \end{aligned}}$$

$dx = \frac{2\, dt}{1+t^2}$

(10)

$$\int \frac{1+\sin x}{1-\cos x}\,dx \underset{\substack{t=\mathrm{tg}\frac{x}{2} \\ dx=\frac{2}{1+t^2}dt}}{=} \int \frac{1+\frac{2\,\mathrm{tg}\frac{x}{2}}{1+\mathrm{tg}^2\frac{x}{2}}}{1-\frac{1-\mathrm{tg}^2\frac{x}{2}}{1+\mathrm{tg}^2\frac{x}{2}}}\,dx = \int \frac{1+\mathrm{tg}^2\frac{x}{2}+2\,\mathrm{tg}\frac{x}{2}}{2\,\mathrm{tg}^2\frac{x}{2}}\,dx =$$

$$= \int \frac{t^2+2t+1}{2t^2(1+t^2)}\,2\,dt = \int \frac{t^2+2t+1}{t^2(t^2+1)}\,dt =$$

$$= \int \left(\frac{2}{t}+\frac{1}{t^2}-\frac{2t}{t^2+1}\right)dt = \ldots$$

(11)

$$\int \frac{1}{2+\sin^2 x}\,dx \underset{\sin^2 x=\frac{1}{1+\mathrm{ctg}^2 x}}{=} \int \frac{1}{2+\frac{1}{1+\mathrm{ctg}^2 x}}\,dx = \int \frac{1+\mathrm{ctg}^2 x}{3+2\,\mathrm{ctg}^2 x}\,dx =$$

$$\underset{\substack{y=\mathrm{ctg}\,x \\ x=\mathrm{arcctg}\,y \\ dx=\frac{1}{1+y^2}dy}}{=} -\int \frac{1+y^2}{3+2y^2}\,\frac{1}{1+y^2}\,dy = -\int \frac{1}{3+2y^2}\,dy =$$

$$= -\frac{1}{3}\int \frac{1}{1+\left(\sqrt{\frac{2}{3}}\,y\right)^2}\,dy =$$

$$= -\frac{1}{\sqrt{6}}\,\mathrm{arctg}\left(\sqrt{\frac{2}{3}}\,\mathrm{ctg}\,x\right)$$

(12)

$$\int \frac{4}{e^{2x}-4}\,dx \underset{\uparrow}{=} \int \frac{4}{t^2-4}\,\frac{dt}{t} = \int\left(-\frac{1}{t}+\frac{1}{2}\,\frac{1}{t+2}+\frac{1}{2}\,\frac{1}{t-2}\right)dt$$

$t = e^x$

$dx = \frac{dt}{t}$

(13)

$$\int x\sqrt{5x+3}\,dx \underset{\uparrow}{=} \int \frac{y^2-3}{5}\,y\,\frac{2}{5}y\,dy = \frac{2}{25}\int (y^4-3y^2)\,dy = \ldots$$

$y = \sqrt{5x+3}$

$x = \frac{y^2-3}{5}$

$dx = \frac{2}{5}y\,dy$

(14)

$$\int \sqrt{\frac{x-3}{x-1}}\,dx \underset{\uparrow}{=} \int \frac{4y^2}{(y^2-1)^2}\,dy = \int\left(\frac{1}{(y+1)^2}-\frac{1}{y+1}+\frac{1}{(y-1)^2}+\frac{1}{y-1}\right)dy$$

$y = \sqrt{\frac{x-3}{x-1}}$

$x = \frac{y^2-3}{y^2-1}$

$dx = \frac{4y}{(y^2-1)^2}\,dy$