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Kalkulus 1. 13–14 Feladatsor megoldások

(1) Alapintegrálokra visszavezethető feladatok

a) $$\int \frac{x^2-7x+8}{x^2}\,dx = \int \left(1-\frac{7}{x}+\frac{8}{x^2}\right)dx = \int \left(1-7\cdot x^{-1}+8x^{-2}\right)dx =$$

$$= x-7\ln|x|+8\,\frac{x^{-1}}{-1}+c = \underline{\underline{x-7\ln|x|-\frac{8}{x}+c}}$$

b) $$\int \frac{x^6}{x^2+1}\,dx \underset{\uparrow}{=} \int \left(x^4-x^2+1-\frac{1}{x^2+1}\right)dx = \underline{\underline{\frac{x^5}{5}-\frac{x^3}{3}+x-\operatorname{arctg} x+c}}$$

polinomosztás: $x^6:(x^2+1)=x^4-x^2+1$

$$\begin{array}{rl} - & x^6+x^4 \\ \hline & -x^4 \\ - & -x^4-x^2 \\ \hline & x^2 \\ - & x^2+1 \\ \hline & -1 \end{array}$$

$$\rightsquigarrow \frac{x^6}{x^2+1} = x^4-x^2+1-\frac{1}{x^2+1}$$

c) $$\int \left(\sqrt{x\sqrt[3]{x}\sqrt[4]{x}}+\frac{1}{\sqrt{x}}\right)dx = \int \left(\left[x\,x^{1/3}x^{1/4}\right]^{1/2}+x^{-1/2}\right)dx = \int \left(\left(x^{19/12}\right)^{1/2}+x^{-1/2}\right)dx$$

$$= \int \left(x^{\frac{19}{24}}+x^{-1/2}\right)dx = \frac{x^{\frac{43}{24}}}{\frac{43}{24}}+\frac{x^{1/2}}{1/2}+c = \underline{\underline{\frac{24}{43}x^{\frac{43}{24}}+2\sqrt{x}+c}}$$

d) $$\int \frac{-2}{5x^2+5}\,dx = -\frac{2}{5}\int \frac{1}{x^2+1}\,dx = \underline{\underline{-\frac{2}{5}\operatorname{arctg} x+c}}$$

e) $\int \frac{2x+3}{x-2}\,dx = \int \frac{2(x-2)+7}{x-2}\,dx = \int \left(2 + \frac{7}{x-2}\right)dx =$

$= 2x + 7\ln|x-2| + c$

f) $\int \frac{1-e^{x}}{1+e^{x/2}}\,dx = \int \frac{(1-e^{x/2})(1+e^{x/2})}{1+e^{x/2}}\,dx = \int (1-e^{x/2})\,dx =$

$= x - \frac{e^{x/2}}{1/2} + c = x - 2e^{x/2} + c$

g) $\int \frac{e^{2x}+1}{e^{3x}}\,dx = \int (e^{-x}+e^{-3x})\,dx = \frac{e^{-x}}{-1} + \frac{e^{-3x}}{-3} + c = -e^{-x} - \frac{e^{-3x}}{3} + c$

h) $\int e^{3x}\,\mathrm{sh}\,5x\,dx = \int e^{3x}\cdot\frac{e^{5x}-e^{-5x}}{2}\,dx = \frac{1}{2}\int (e^{8x}-e^{-2x})\,dx =$

$\uparrow$

eml: $\mathrm{sh}\,\alpha = \frac{e^{\alpha}-e^{-\alpha}}{2}$

$= \frac{1}{2}\left(\frac{e^{8x}}{8} - \frac{e^{-2x}}{-2}\right) + c = \frac{e^{8x}}{16} + \frac{e^{-2x}}{4} + c$

megj: lehetne parciális integrálással is, de úgy hosszabb.

i) $\int \frac{1}{3x^2+6}\,dx = \frac{1}{6}\int \frac{1}{\frac{x^2}{2}+1}\,dx = \frac{1}{6}\int \frac{1}{\left(\frac{x}{\sqrt{2}}\right)^2+1}\,dx =$

$= \frac{1}{6}\,\frac{\mathrm{arctg}\left(\frac{x}{\sqrt{2}}\right)}{1/\sqrt{2}} + c = \frac{\sqrt{2}}{6}\,\mathrm{arctg}\left(\frac{x}{\sqrt{2}}\right) + c$

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j) $$\int \sqrt{3x-1}\,dx = \int (3x-1)^{1/2}\,dx = \frac{(3x-1)^{3/2}}{3/2}\cdot\frac{1}{3}+c =$$

$$= \frac{2}{9}(3x-1)^{3/2}+c$$

k) $$\int \frac{1}{\sqrt[3]{3x-2}}\,dx = \int (3x-2)^{-1/3}\,dx = \frac{(3x-2)^{2/3}}{\frac{2}{3}\cdot 3}+c =$$

$$= \frac{(3x-2)^{2/3}}{2}+c$$

(2) Teljes négyzetté alakítással alapintegrálokra visszavezethető feladatok

a) $$\int \frac{dx}{x^2+2x+6} = \int \frac{dx}{(x+1)^2+5} = \frac{1}{5}\int \frac{1}{\left(\frac{x+1}{\sqrt{5}}\right)^2+1}\,dx = \frac{1}{5}\,\frac{\operatorname{arctg}\left(\frac{x+1}{\sqrt{5}}\right)}{\frac{1}{\sqrt{5}}}+c \;(=)$$

a cél: $$\boxed{\int \frac{1}{(ax+b)^2+1}\,dx = \frac{\operatorname{arctg}(ax+b)}{a}+c}$$ használata

$$(=)\; \frac{1}{\sqrt{5}}\operatorname{arctg}\left(\frac{x+1}{\sqrt{5}}\right)+c$$

b) $$\int \frac{dx}{x^2-6x+1} = \int \frac{dx}{(x-3)^2-8} = -\frac{1}{8}\int \frac{dx}{1-\left(\frac{x-3}{\sqrt{8}}\right)^2} = -\frac{1}{8}\begin{cases} \frac{\operatorname{arth}\left(\frac{x-3}{\sqrt{8}}\right)}{\frac{1}{\sqrt{8}}}+c, & \text{ha } \left|\frac{x-3}{\sqrt{8}}\right|<1 \\ \frac{\operatorname{arcth}\left(\frac{x-3}{\sqrt{8}}\right)}{\frac{1}{\sqrt{8}}}+c, & \text{ha } \left|\frac{x-3}{\sqrt{8}}\right|>1 \end{cases}$$

cél: $$\boxed{\int \frac{1}{1-x^2}\,dx = \begin{cases} \operatorname{arth} x, & \text{ha } |x|<1 \\ \operatorname{arcth} x, & \text{ha } |x|>1 \end{cases}}$$ használata

$$+\quad \int f(x)\,dx = F(x)+c \;\Rightarrow\; \int f(ax+b)\,dx = \frac{F(ax+b)}{a}+c$$

c)

$$\int \frac{dx}{(2x^2-12x+23)} = \frac{1}{2}\int \frac{dx}{x^2-6x+\frac{23}{2}} = \frac{1}{2}\int \frac{dx}{(x-3)^2+\frac{5}{2}} =$$

$$= \frac{1}{2}\cdot\frac{2}{5}\int \frac{dx}{\left[\frac{x-3}{\sqrt{\frac{2}{5}}}\right]^2+1} \underset{P}{=} \frac{1}{5}\int \frac{dx}{\left(\frac{\sqrt{5}}{\sqrt{2}}x-\frac{3\sqrt{5}}{\sqrt{2}}\right)^2+1} \; (=)$$

$$\int \frac{1}{1+x^2}\,dx = \operatorname{arctg} x + c$$

$$(=) \; \frac{1}{5}\,\frac{\operatorname{arctg}\left(\frac{\sqrt{5}}{\sqrt{2}}x-\frac{3\sqrt{5}}{\sqrt{2}}\right)}{\frac{\sqrt{5}}{\sqrt{2}}} + c = \frac{\sqrt{2}}{5\sqrt{5}}\operatorname{arctg}\left(\sqrt{\frac{5}{2}}x-\frac{3\sqrt{5}}{\sqrt{2}}\right)+c$$

d)

$$\int \frac{dx}{\sqrt{x^2-4x+40}} = \int \frac{dx}{\sqrt{(x-2)^2+36}} = \frac{1}{6}\int \frac{dx}{\sqrt{\left(\frac{x-2}{6}\right)^2+1}} =$$

$$\underset{P}{=} \frac{1}{6}\,\frac{\operatorname{arsh}\left(\frac{x-2}{6}\right)}{\frac{1}{6}} + c = \operatorname{arsh}\frac{x-2}{6}+c$$

$$\boxed{\int \frac{1}{\sqrt{x^2+1}}\,dx = \operatorname{arsh} x + c}$$

e)

$$\int \frac{dx}{\sqrt{x^2+6x}} = \int \frac{dx}{\sqrt{(x+3)^2-9}} = \frac{1}{3}\int \frac{dx}{\sqrt{\left(\frac{x-3}{3}\right)^2-1}} =$$

$$\underset{P}{=} \frac{1}{3}\,\frac{\operatorname{arch}\frac{x-3}{3}}{\frac{1}{3}} + c = \operatorname{arch}\frac{x-3}{3}+c$$

$$\boxed{\int \frac{1}{\sqrt{x^2-1}}\,dx = \operatorname{arch} x + c \quad x>1}$$

3)

③ Trigonometrikus szorzatintegrálok

Leggyakrabban használt formulák levezetése az Euler-alakból:

$$e^{i(\alpha+\beta)} = \underline{\cos(\alpha+\beta)+i\sin(\alpha+\beta)} = e^{i\alpha}\cdot e^{i\beta} =$$

$$= (\cos\alpha + i\sin\alpha)\cdot(\cos\beta + i\sin\beta) =$$

$$= \underline{\cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)}$$

$$\Rightarrow \boxed{\cos(\alpha+\beta) = \cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta} \quad (1.)$$

$$\boxed{\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta} \quad (2)$$

β helyett $(-\beta)$-t írva és kihasználva, hogy $\cos(-\beta)=\cos\beta$

$\sin(-\beta)=-\sin\beta$

$$\Rightarrow \boxed{\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\cdot\sin\beta} \quad (3.)$$

$$\boxed{\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta} \quad (4.)$$

Spec: (1)-ben $\beta=\alpha \Rightarrow \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$+ \quad \cos^2\alpha + \sin^2\alpha = 1$

$$\boxed{\cos^2\alpha = \frac{1+\cos 2\alpha}{2}} \quad (5)$$

$$\boxed{\sin^2\alpha = \frac{1-\cos 2\alpha}{2}} \quad (6.)$$

$(1.)+(3.) \Rightarrow \quad \boxed{\cos\alpha\cdot\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta)+\cos(\alpha-\beta)\right)} \quad (7.)$

$(2.)+(4.) \Rightarrow \quad \boxed{\sin\alpha\cdot\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta)+\sin(\alpha-\beta)\right)} \quad (8.)$

$(3.)-(1.) \Rightarrow \quad \boxed{\sin\alpha\cdot\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right)} \quad (9.)$

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a) $\int \cos^4 x\,dx \underset{\uparrow}{=} \frac{1}{4}\int\left(\frac{3}{2}+\cos 2x+\frac{1}{2}\cos 4x\right)dx \;\;(=)$

$\cos^4 x = (\cos^2 x)^2 \underset{\substack{\uparrow\\(5.)}}{=} \left(\frac{1+\cos 2x}{2}\right)^2 = \frac{1}{4}\left(1+2\cos 2x+\cos^2 2x\right) \underset{\substack{\uparrow\\(5.)}}{=} \frac{1}{4}\left(1+\cos 2x+\frac{1+\cos 4x}{2}\right)$

$(=)\; \frac{1}{4}\left(\frac{3}{2}x+\frac{\sin 2x}{2}+\frac{1}{2}\frac{\sin 4x}{4}\right)+C = \underline{\underline{\frac{3}{8}x+\frac{\sin 2x}{8}+\frac{\sin 4x}{32}+C}}$

b) $\int \sin^2(1-2x)\,dx \underset{\substack{\uparrow\\(6.)}}{=} \int \frac{1-\cos(2-4x)}{2}\,dx = \frac{1}{2}x-\frac{1}{2}\left(\frac{\sin(2-4x)}{-4}\right)+C =$

$= \underline{\underline{\frac{1}{2}x+\frac{\sin(2-4x)}{8}+C}}$

c) $\int \sin 2x\cdot\cos 5x\,dx \underset{\substack{\uparrow\\(8.)}}{=} \frac{1}{2}\int\left(\sin(2x+5x)+\sin(2x-5x)\right)dx \underset{\substack{\uparrow\\\sin(-\alpha)=-\sin\alpha}}{=}$

$= \frac{1}{2}\int\left(\sin 7x-\sin 3x\right)dx = \frac{1}{2}\left(\frac{-\cos 7x}{7}-\frac{-\cos 3x}{3}\right)+C = \underline{\underline{-\frac{\cos 7x}{14}+\frac{\cos 3x}{6}+C}}$

4)

(5) $$\boxed{\int f'(x)\cdot f^{\alpha}(x)\,dx = \frac{f^{\alpha+1}(x)}{\alpha+1} + c \qquad \alpha \neq -1}$$ Lipus

a) $$\int \frac{2x-5}{\sqrt[3]{(x^2-5x+10)^7}}\,dx = \int (2x-5)\cdot(x^2-5x+10)^{-7/3}\,dx \underset{\uparrow}{=}$$

$(x^2-5x+10)' = 2x-5$

$$= \frac{(x^2-5x+10)^{-4/3}}{-4/3} + c = -\frac{3}{4}\,\frac{1}{\sqrt[3]{(x^2-5x+10)^4}} + c$$

b) $$\int \sin^3 x\cdot\cos x\,dx \underset{\uparrow}{=} \frac{\sin^4 x}{4} + c$$

$(\sin x)' = \cos x$

c) $$\int \frac{\operatorname{tg} x}{\cos^2 x}\,dx = \int \frac{1}{\cos^2 x}\cdot\operatorname{tg} x\,dx \underset{\uparrow}{=} \frac{\operatorname{tg}^2 x}{2} + c$$

$(\operatorname{tg} x)' = \frac{1}{\cos^2 x}$

d) $$\int \cos^5 x\,dx = \int \cos x\cdot\cos^4 x\,dx \underset{\uparrow}{=} \int (\cos x - 2\cos x\sin^2 x + \cos x\sin^4 x)\,dx \;(=)$$

$\cos^4 x = (\cos^2 x)^2 = (1-\sin^2 x)^2 = 1 - 2\sin^2 x + \sin^4 x$

$$\underset{\uparrow}{(=)}\; \sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + c$$

$(\sin x)' = \cos x$

e)
$$\int \sin^3 x \, dx = \int \sin x \cdot \sin^2 x \, dx \underset{\cap}{=} \int (\sin x - \sin x \cdot \cos^2 x)\, dx \;(=)$$

$$\sin^2 x = 1 - \cos^2 x$$

$$\underset{\cap}{(=)} -\cos x + \frac{\cos^3 x}{3} + c$$

$$(\cos x)' = -\sin x$$

f)
$$\int \frac{\cos x}{\sin^2 x}\, dx = \int \cos x \sin^{-2} x \, dx \underset{\cap}{=} \frac{\sin^{-1} x}{-1} + c = -\frac{1}{\sin x} + c$$

$$(\sin x)' = \cos x$$

g)
$$\int \frac{\operatorname{ch} x}{\sqrt[3]{\operatorname{sh} x}}\, dx = \int \operatorname{ch} x \cdot \operatorname{sh}^{-1/3} x \, dx \underset{\cap}{=} \frac{\operatorname{sh}^{2/3} x}{2/3} + c = \frac{3}{2} \operatorname{sh}^{2/3} x + c$$

$$(\operatorname{sh} x)' = \operatorname{ch} x$$

h)
$$\int \frac{\sqrt{\ln^3 x}}{x}\, dx = \int \frac{1}{x} \ln^{3/2} x \, dx \underset{\cap}{=} \frac{\ln^{5/2} x}{5/2} + c = \frac{2}{5} \ln^{5/2} x + c$$

$$(\ln x)' = \frac{1}{x}$$

i)
$$\int \frac{e^x}{e^{2x} + 2e^x + 1}\, dx = \int \frac{e^x}{(e^x+1)^2}\, dx = \int e^x \cdot (e^x+1)^{-2}\, dx \underset{\cap}{=}$$

$$(e^x+1)' = e^x$$

$$= \frac{(e^x+1)^{-1}}{-1} + c = -\frac{1}{e^x+1} + c$$

5)

(5) $\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C \qquad f(x) \neq 0}$ típus

a) $\int \frac{5x^2}{5-4x^3}\,dx = -\frac{5}{12}\int \frac{-12x^2}{5-4x^3}\,dx = -\frac{5}{12}\ln|5-4x^3| + C$

$(5-4x^3)' = -12x^2$

b) $\int \operatorname{tg} x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{-\sin x}{\cos x}\,dx = -\ln|\cos x| + C$

$(\cos x)' = -\sin x$

c) $\int \frac{1}{x\ln x}\,dx = \int \frac{\frac{1}{x}}{\ln x}\,dx = \ln|\ln x| + C$

$(\ln x)' = \frac{1}{x}$

d) $\int \frac{9x^2}{4x^3+5}\,dx = \frac{9}{12}\int \frac{12x^2}{4x^3+5}\,dx = \frac{9}{12}\ln|4x^3+5| + C$

$(4x^3+5)' = 12x^2$

e) $\int \frac{e^{3x}}{e^{3x}+6}\,dx = \frac{1}{3}\int \frac{3e^{3x}}{e^{3x}+6}\,dx = \frac{1}{3}\ln(e^{3x}+6) + C$

$(e^{3x}+6)' = 3e^{3x}$

f) $\int \frac{1}{(1+x^2)\operatorname{arctg} x}\,dx = \int \frac{\frac{1}{1+x^2}}{\operatorname{arctg} x}\,dx = \ln|\operatorname{arctg} x| + C$

$(\operatorname{arctg} x)' = \frac{1}{1+x^2}$

⑥ Ha $\boxed{F'(x) = f(x) \text{, akkor } \int f(g(x))\, g'(x)\, dx = F(g(x)) + c}$

a) $\displaystyle\int \frac{e^{tg x}}{\cos^2 x}\, dx = \int e^{tg x} \cdot \frac{1}{\cos^2 x}\, dx \underset{\uparrow}{=} \underline{\underline{e^{tg x} + c}}$

$(tg\, x)' = \dfrac{1}{\cos^2 x}$

$\int e^x\, dx = e^x + c$

b) $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\, dx = \int (1-e^{2x})^{-1/2} e^x\, dx \underset{\uparrow}{=} \underline{\underline{\arcsin e^x + c}}$

$(e^x)' = e^x$

$\displaystyle\int (1-x^2)^{-1/2}\, dx = \int \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin x + c$

c) $\displaystyle\int x \sin x^2\, dx = \frac{1}{2} \int 2x \cdot \sin x^2\, dx \underset{\uparrow}{=} \underline{\underline{-\frac{1}{2} \cos x^2 + c}}$

$(x^2)' = 2x$

$\int \sin x\, dx = -\cos x + c$

d) $\displaystyle\int \frac{sh \sqrt{x}}{\sqrt{x}}\, dx = 2 \int \frac{1}{2} x^{-1/2}\, sh \sqrt{x}\, dx \underset{\uparrow}{=} \underline{\underline{2\, ch \sqrt{x} + c}}$

$(\sqrt{x})' = (x^{1/2})' = \frac{1}{2} x^{-1/2}$

$\int sh\, x\, dx = ch\, x + c$

e) $\displaystyle\int \frac{\ln \sqrt{x}}{x}\, dx = \frac{1}{2} \int \frac{1}{x} \ln x\, dx \underset{\uparrow}{=} \frac{1}{2}\, \frac{\ln^2 x}{2} + c = \underline{\underline{\frac{1}{4} \ln^2 x + c}}$

$\int f' \cdot f = \dfrac{f^2}{2} + c$

6/

(7) Parciális integrálás, azaz $\boxed{\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx}$

a) $$\int x^2 \sin 2x\,dx = -x^2\frac{\cos 2x}{2} + \int \frac{\cos 2x}{2}\cdot 2x\,dx \;\ominus$$

$f'(x) = \sin 2x \rightsquigarrow f(x) = -\frac{\cos 2x}{2}$

$g(x) = x^2 \rightsquigarrow g'(x) = 2x$

$$\ominus\; -x^2\frac{\cos 2x}{2} + \int x\cos 2x\,dx = -x^2\frac{\cos 2x}{2} + x\cdot\frac{\sin 2x}{2} - \int\frac{\sin 2x}{2}\,dx =$$

$f'(x) = \cos 2x \rightsquigarrow f(x) = \frac{\sin 2x}{2}$

$g(x) = x \rightsquigarrow g'(x) = 1$

$$= -x^2\frac{\cos 2x}{2} + x\frac{\sin 2x}{2} + \frac{\cos 2x}{4} + C$$

b) $$I := \int e^{2x}\cos 3x\,dx = \frac{e^{2x}}{2}\cdot\cos 3x + \int\frac{e^{2x}}{2}\cdot 3\sin 3x\,dx \;\ominus$$

$f'(x) = e^{2x} \rightsquigarrow f(x) = \frac{e^{2x}}{2}$

$g(x) = \cos 3x \to g'(x) = -3\sin 3x$

$$\ominus\; \frac{e^{2x}}{2}\cos 3x + \frac{3}{2}\int e^{2x}\sin 3x\,dx = \frac{e^{2x}}{2}\cos 3x + \frac{3}{2}\left[\frac{e^{2x}}{2}\sin 3x - \frac{3}{2}\int e^{2x}\cos 3x\,dx\right]$$

$f'(x) = e^{2x} \rightsquigarrow f(x) = \frac{e^{2x}}{2}$

$g(x) = \sin 3x \rightsquigarrow g'(x) = 3\cos 3x$

$$= \frac{e^{2x}}{2}\cos 3x + \frac{3}{4}e^{2x}\sin 3x - \frac{9}{4}I$$

rendezve: $\frac{13}{4}I = \frac{e^{2x}}{2}\cos 3x + \frac{3}{4}e^{2x}\sin 3x$

$$\Rightarrow I = \frac{4}{13}\left(\frac{e^{2x}}{2}\cos 3x + \frac{3}{4}e^{2x}\sin 3x\right) + C$$

c) $$\int e^x \sin^2 x \, dx \underset{\uparrow}{=} \frac{1}{2}\int \left(e^x - e^x \cos 2x\right) dx = \frac{1}{2} e^x - \frac{1}{2}\underbrace{\int e^x \cos 2x \, dx}_{I}$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

jelölje: $$I = \int e^x \cos 2x \, dx \underset{\uparrow}{=} e^x \cos 2x + 2\int e^x \sin 2x \, dx \ (=)$$

$$f'(x) = e^x \rightsquigarrow f(x) = e^x$$

$$g'(x) = \cos 2x \rightsquigarrow g'(x) = -2\sin 2x$$

$$\underset{\uparrow}{(=)} \ e^x \cos 2x + 2\left(e^x \sin 2x - \int e^x \cdot 2\cos 2x \, dx\right) =$$

$$f'(x) = e^x \rightsquigarrow f(x) = e^x$$

$$g(x) = \sin 2x \rightsquigarrow g'(x) = 2\cos 2x$$

$$= e^x \cos 2x + 2e^x \sin 2x - 4I \rightsquigarrow 5I = e^x \cos 2x + 2e^x \sin 2x$$

$$\Rightarrow I = \frac{1}{5}\left(e^x \cos 2x + 2e^x \sin 2x\right) + c$$

<u>ezzel:</u> $$\int e^x \sin^2 x \, dx = \frac{1}{2} e^x - \frac{1}{10}\left(e^x \cos 2x + 2e^x \sin 2x\right) + c$$

d) $$\int x^2 e^{-2x} \, dx \underset{\uparrow}{=} -x^2 \frac{e^{-2x}}{2} + \int e^{-2x} \cdot x \, dx \ \underset{\uparrow}{(=)}$$

$$f'(x) = e^{-2x} \rightsquigarrow f(x) = -\frac{e^{-2x}}{2} \qquad f'(x) = e^{-2x} \rightsquigarrow f(x) = -\frac{e^{-2x}}{2}$$

$$g(x) = x^2 \rightsquigarrow g'(x) = 2x \qquad g(x) = x \rightsquigarrow g'(x) = 1$$

$$(=) -x^2 \frac{e^{-2x}}{2} - x\frac{e^{-2x}}{2} + \int \frac{e^{-2x}}{2} dx = -x^2 \frac{e^{-2x}}{2} - x\frac{e^{-2x}}{2} - \frac{e^{-2x}}{4} + c$$

7)

e) $\int x \arctan x \, dx \underset{\uparrow}{=} \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx = \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2+1-1}{1+x^2}\,dx$

$f'(x) = x \rightsquigarrow f(x) = \frac{x^2}{2}$

$g(x) = \arctan x \rightsquigarrow g'(x) = \frac{1}{1+x^2}$

$= \frac{x^2}{2}\arctan x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right)dx = \underline{\underline{\frac{x^2}{2}\arctan x - \frac{1}{2}\left(x - \arctan x\right) + c}}$

f) $\int \arcsin x \, dx = \int 1 \cdot \arcsin x \, dx \underset{\uparrow}{=} x \arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,dx \;\ominus$

$f'(x) = 1 \rightsquigarrow f(x) = x$

$g(x) = \arcsin x \rightsquigarrow g'(x) = \frac{1}{\sqrt{1-x^2}}$

$\ominus\; x \arcsin x + \frac{1}{2} \underbrace{\int (-2x)\cdot(1-x^2)^{-1/2}\,dx}_{\int f' \cdot f^{-1/2} = \frac{f^{1/2}}{1/2} + c} = x \arcsin x + \frac{1}{2}\,\frac{(1-x^2)^{1/2}}{1/2} + c =$

$= \underline{\underline{x \arcsin x + \sqrt{1-x^2} + c}}$

g) $\int \ln^3 x \, dx = \int 1 \cdot \ln^3 x \, dx \underset{\uparrow}{=} x \ln^3 x - \int x \cdot 3\ln^2 x \cdot \frac{1}{x}\,dx \;\ominus$

$f'(x) = 1 \rightsquigarrow f(x) = x$

$g(x) = \ln^3 x \quad g'(x) = 3 \cdot \ln^2 x \cdot \frac{1}{x}$

$\ominus\; x \ln^3 x - 3\int \ln^2 x \, dx \underset{\uparrow}{=} x \ln^3 x - 3\left(x \ln^2 x - \int x \cdot 2 \cdot \ln x \cdot \frac{1}{x}\,dx\right) \boxed{=}$

$f'(x) = 1 \rightsquigarrow f(x) = x$

$g(x) = \ln^2 x \rightsquigarrow g'(x) = 2\ln x \cdot \frac{1}{x}$

$\boxed{=} \; x\ln^3 x - 3\left(x\ln^2 x - 2\int \ln x\,dx\right) = x\ln^3 x - 3x\ln^2 x + 6\int 1\cdot\ln x\,dx =$

$\underset{P}{=} x\ln^3 x - 3x\ln^2 x + 6\left(x\ln x - \underbrace{\int x\cdot\frac{1}{x}\,dx}_{\int dx = x + c}\right) =$

$f'(x) = 1 \rightsquigarrow f(x) = x$

$g(x) = \ln x \rightsquigarrow g'(x) = \frac{1}{x}$

$= x\ln^3 x - 3x\ln^2 x + 6x\ln x - 6x + c$

h) $\int x\,\mathrm{sh}\,x\,dx \underset{P}{=} x\,\mathrm{ch}\,x - \int \mathrm{ch}\,x\,dx = x\,\mathrm{ch}\,x - \mathrm{sh}\,x + c$

$f'(x) = \mathrm{sh}\,x \rightsquigarrow f(x) = \mathrm{ch}\,x$

$g(x) = x \rightsquigarrow g'(x) = 1$

i) $\int \frac{\ln\cos x}{\cos^2 x}\,dx \underset{P}{=} \mathrm{tg}\,x\cdot\ln\cos x - \int \mathrm{tg}^2 x\,dx$ ⊜

$f'(x) = \frac{1}{\cos^2 x} \rightsquigarrow f(x) = \mathrm{tg}\,x$

$g(x) = \ln\cos x \rightsquigarrow g'(x) = \frac{1}{\cos x}\cdot(-\sin x) = -\mathrm{tg}\,x$

⊜ $\mathrm{tg}\,x\cdot\ln\cos x - \underbrace{\int \frac{\sin^2 x}{\cos^2 x}\,dx} = \mathrm{tg}\,x\cdot\ln\cos x - \mathrm{tg}\,x + x + c$

$\int \frac{1-\cos^2 x}{\cos^2 x}\,dx = \int\left(\frac{1}{\cos^2 x} - 1\right)dx = \mathrm{tg}\,x - x + c$

8)

i) $$\int \frac{x^2 \operatorname{arctg} x}{1+x^2}\,dx = \int \frac{(1+x^2-1)\operatorname{arctg} x}{1+x^2}\,dx = \int \left(\operatorname{arctg} x - \frac{\operatorname{arctg} x}{1+x^2}\right)dx$$

$$= \int \operatorname{arctg} x\,dx - \underbrace{\int \frac{1}{1+x^2}\cdot \operatorname{arctg} x\,dx}_{\int f'\cdot f = \frac{f^2}{2}+c} =$$

$$= \int 1\cdot \operatorname{arctg} x\,dx - \frac{(\operatorname{arctg} x)^2}{2} + c \underset{\uparrow}{=} x \operatorname{arctg} x - \int \frac{x}{1+x^2}\,dx - \frac{(\operatorname{arctg} x)^2}{2} + c \;\ominus$$

$f'(x) = 1 \leadsto f(x) = x$

$g(x) = \operatorname{arctg} x \leadsto g'(x) = \frac{1}{1+x^2}$

$$\ominus\; x \operatorname{arctg} x - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx - \frac{(\operatorname{arctg} x)^2}{2} =$$

$\uparrow$

$$\int \frac{f'}{f} = \ln|f| + c$$

$$= x \operatorname{arctg} x - \frac{1}{2}\ln(1+x^2) - \frac{(\operatorname{arctg} x)^2}{2} + c$$

k) $$I := \int \cos x \cdot \operatorname{sh} x\,dx \underset{\uparrow}{=} \operatorname{ch} x \cdot \cos x + \int \operatorname{ch} x \cdot \sin x\,dx \underset{\uparrow}{=}$$

$f'(x) = \operatorname{sh} x \leadsto f(x) = \operatorname{ch} x$

$g(x) = \cos x \qquad g'(x) = -\sin x$

$f'(x) = \operatorname{ch} x \leadsto f(x) = \operatorname{sh} x$

$g(x) = \sin x \leadsto g'(x) = \cos x$

$$= \operatorname{ch} x \cos x + \operatorname{sh} x \cdot \sin x - \underbrace{\int \operatorname{sh} x \cdot \cos x\,dx}_{I}$$

$$\Rightarrow 2I = \operatorname{ch} x \cos x + \operatorname{sh} x \sin x$$

$$\hookrightarrow \quad I = \frac{1}{2}(\operatorname{ch} x \cos x + \operatorname{sh} x \sin x) + c$$

(l) $$\int \frac{xe^x}{(x+1)^2}\,dx \underset{P}{=} -\frac{xe^x}{x+1} + \int \frac{\overbrace{e^x+xe^x}^{e^x(1+x)}}{x+1}\,dx \;=$$

$$f'(x)=\frac{1}{(x+1)^2} \rightsquigarrow f(x)=-\frac{1}{x+1}$$

$$g(x)=xe^x \rightsquigarrow g'(x)=e^x+xe^x$$

$$= -\frac{xe^x}{x+1}+\int e^x\,dx = \underline{\underline{-\frac{xe^x}{x+1}+e^x+c}}$$

(8) Helyettesítéses integrálás

a) $$\int \sqrt{5+3x^2}\,dx \underset{P}{=} \sqrt{5}\int \sqrt{1+\left(\frac{\sqrt{3}x}{\sqrt{5}}\right)^2}\,dx \;\underset{P}{=}$$

~~az megoldható helyettesítés~~
~~nélkül is, hanem tudjuk, hogy~~

mivel
$ch^2t - sh^2t = 1$
$\hookrightarrow ch^2t = 1+sh^2t$

ezért próbálkozunk

$$sh\,t = \frac{\sqrt{3}x}{\sqrt{5}}$$ helyettesítéssel

$$\Rightarrow x=\frac{\sqrt{5}}{\sqrt{3}}\,sh\,t \Rightarrow \frac{dx}{dt}=\frac{\sqrt{5}}{\sqrt{3}}\,ch\,t$$

$$\hookrightarrow dx = \frac{\sqrt{5}}{\sqrt{3}}\,ch\,t\,dt$$

$$= \sqrt{5}\int\sqrt{1+sh^2t}\;\frac{\sqrt{5}}{\sqrt{3}}\,ch\,t\,dt = \frac{5}{\sqrt{3}}\int ch^2t\,dt$$

9)

emlékeztetünk, hogy mivel $ch\,t = \dfrac{e^t+e^{-t}}{2}$ és $sh\,t = \dfrac{e^t-e^{-t}}{2}$,

ezért egyrészt $ch^2t - sh^2t = 1$,

másrészt $ch^2t + sh^2t = ch\,2t$.

Innen $\boxed{ch^2t = \dfrac{ch\,2t+1}{2}}$ és $\boxed{sh^2t = \dfrac{ch\,2t-1}{2}}$

Vagyis: $$\int ch^2t\,dt = \int \frac{ch\,2t+1}{2}\,dt = \frac{sh\,2t}{4} + \frac{1}{2}t + c.$$

$$\Rightarrow \int\sqrt{5+3x^2}\,dx = \frac{5}{\sqrt{3}}\int ch^2t\,dt = \frac{5}{\sqrt{3}}\left(\frac{sh\,2t}{4}+\frac{1}{2}t\right)+c$$

visszahelyettesítés: $t = arsh\,\dfrac{\sqrt{3}x}{\sqrt{5}}$

Mivel $$sh\,2t = 2\,sh\,t\,ch\,t \underset{ch\,t=\sqrt{1+sh^2t}}{=} 2\,sh\,t\sqrt{1+sh^2t} = 2\left(sh\,arsh\frac{\sqrt{3}x}{\sqrt{5}}\right)\cdot\sqrt{1+sh^2\left(arsh\frac{\sqrt{3}x}{\sqrt{5}}\right)}$$

$$= 2\,\frac{\sqrt{3}x}{\sqrt{5}}\cdot\sqrt{1+\frac{3x^2}{5}}$$

$$\Rightarrow \int\sqrt{5+3x^2}\,dx = \frac{5}{\sqrt{3}}\left(\frac{1}{2}\,\frac{\sqrt{3}x}{\sqrt{5}}\sqrt{1+\frac{3x^2}{5}} + \frac{1}{2}\,arsh\,\frac{\sqrt{3}x}{\sqrt{5}}\right)+c$$

b) $\int \sqrt{(x^2-1)^3}\,dx = \int \left(\sqrt{x^2-1}\right)^3 dx \qquad (|x| \geq 1)$

Jó lenne $\sqrt{\ }$-től „megszabadulni".

$ch^2 t - sh^2 t = 1 \rightsquigarrow sh^2 t = ch^2 t - 1$

Próbálkozunk $x = ch\,t$ helyettesítéssel

$\hookrightarrow \frac{dx}{dt} = sh\,t \rightsquigarrow dx = sh\,t\,dt$

az integrandus:

$$\left(\sqrt{x^2-1}\right)^3 \underset{x = ch\,t}{=} \left(\sqrt{ch^2 t - 1}\right)^3 = sh^3 t$$

$\Rightarrow \int \left(\sqrt{x^2-1}\right)^3 dx = \int sh^3 t \cdot sh\,t\,dt = \int sh^4 t\,dt$

Mivel $sh^2 t = \frac{ch\,2t - 1}{2}$

$\hookrightarrow sh^4 t = \left(\frac{ch\,2t-1}{2}\right)^2 = \frac{1}{4}\left(ch^2 2t - 2ch\,2t + 1\right) \underset{ch^2 \alpha = \frac{ch\,2\alpha+1}{2}}{=}$

$= \frac{1}{4}\left(\frac{ch\,4t+1}{2} - 2\,ch\,2t + 1\right) =$

$= \frac{1}{4}\left(\frac{1}{2}ch\,4t + \frac{3}{2} - 2\,ch\,2t\right)$

10)

$$\Rightarrow \int sh^4 t\, dt = \frac{1}{8}\int ch\, 4t\, dt + \frac{3}{8}\int dt - \frac{1}{2}\int ch\, 2t\, dt =$$

$$= \frac{1}{8}\,\frac{sh\, 4t}{4} + \frac{3}{8}t - \frac{1}{2}\,\frac{sh\, 2t}{2} + C =$$

$$= \frac{1}{32} sh\, 4t + \frac{3}{8} t - \frac{1}{4} sh\, 2t + C$$

visszahelyettesítés: $t = arch\, x$

$$\hookrightarrow \int (\sqrt{x^2-1})^3 dx = \underline{\underline{\frac{1}{32} sh(4\, arch\, x) + \frac{3}{8} arch\, x - \frac{1}{4} sh(2\, arch\, x) + C}}$$

lehetne szebb alakra hozni, de ettől most eltekintünk :)

c)

$$\int \frac{e^{2x}}{e^x+1}\, dx \underset{\uparrow}{=} \int \frac{t^2}{t+1}\cdot\frac{1}{t}\, dt = \int \frac{t}{t+1}\, dt = \int \frac{t+1-1}{t+1}\, dt =$$

$t := e^x \rightsquigarrow x = \ln t$

$$\frac{dx}{dt} = \frac{1}{t} \rightsquigarrow dx = \frac{1}{t}\, dt$$

$$= \int \left(1 - \frac{1}{t+1}\right) dt = t - \ln|t+1| + C \underset{\substack{\uparrow \\ t=e^x}}{=} \underline{\underline{e^x - \ln(e^x+1) + C}}$$

d) $$\int \sqrt{3-2x-x^2}\,dx \underset{\uparrow}{=} \int \sqrt{4-(x+1)^2}\,dx = 2\int \sqrt{1-\left(\frac{x+1}{2}\right)^2}\,dx$$

$$3-2x-x^2 = 3-\left((x+1)^2-1\right) = 4-(x+1)^2$$

$\uparrow$

$-1 \le \frac{x+1}{2} \le 1$ szükséges!

Megint szeretnénk a gyöktől megszabadulni:

$$\cos^2 t + \sin^2 t = 1 \leadsto \cos^2 t = 1-\sin^2 t$$

$\Rightarrow$ Próbálkozunk $\sin t = \frac{x+1}{2}$ helyettesítéssel!

↳ $x = 2\sin t - 1$

↳ $\frac{dx}{dt} = 2\cos t \Rightarrow dx = 2\cos t\,dt$

$$\Rightarrow 2\int \sqrt{1-\left(\frac{x+1}{2}\right)^2}\,dx = 2\int \sqrt{1-\sin^2 t}\,2\cos t\,dt = 4\int \cos^2 t\,dt =$$

$$\underset{\uparrow}{=} 2\int (1+\cos 2t)\,dt = 2\left(t + \frac{\sin 2t}{2}\right) + C = 2t + \sin 2t + C$$

$\cos^2 t = \frac{1+\cos 2t}{2}$

<u>visszahelyettesítés:</u> $t = \arcsin\frac{x+1}{2}$

$$\sin 2t = 2\sin t\cos t = 2\sin t\sqrt{1-\sin^2 t} = 2\sin\left(\arcsin\frac{x+1}{2}\right)\sqrt{1-\sin^2\left(\arcsin\frac{x+1}{2}\right)}$$

$$= 2\,\frac{x+1}{2}\sqrt{1-\left(\frac{x+1}{2}\right)^2}$$

$\Rightarrow$
$$\int \sqrt{3-2x-x^2}\,dx = 2\arcsin\frac{x+1}{2} + (x+1)\sqrt{1-\left(\frac{x+1}{2}\right)^2} + C$$

11/

e) $\displaystyle\int \frac{dx}{\sqrt{x}+\sqrt[4]{x}} \underset{\uparrow}{=} \int \frac{4t^3\,dt}{t^2+t} = \int \frac{4t^2}{t^2+1}\,dt \ \textcircled{=}$

$t := \sqrt[4]{x} \rightsquigarrow x = t^4 \rightsquigarrow \frac{dx}{dt} = 4t^3 \Rightarrow dx = 4t^3\,dt$

$\sqrt{x} = (\sqrt[4]{x})^2 = t^2$

$\textcircled{=}\ 4\displaystyle\int \frac{t^2+1-1}{t^2+1}\,dt = 4\int\left(1-\frac{1}{t^2+1}\right)dt = 4(t-\operatorname{arctg} t)+c \underset{\uparrow}{=}$

$t = \sqrt[4]{x}$

$= 4\left(\sqrt[4]{x} - \operatorname{arctg}\sqrt[4]{x}\right)+c$

f) $\displaystyle\int \frac{dx}{\sin x+\cos x} \underset{\uparrow}{=} \int \frac{\frac{2\,dt}{1+t^2}}{\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} = \int \frac{2\,dt}{2t+1-t^2}$

előadáson vettük: $t := \operatorname{tg}\frac{x}{2} \Longrightarrow \sin x = \frac{2t}{1+t^2}$

$\cos x = \frac{1-t^2}{1+t^2}$

$dx = \frac{2\,dt}{1+t^2}$

$-t^2+2t+1=0 \Rightarrow D = 4+4 = 8$ (diszkrimináns) $\rightarrow t_{1,2} = \frac{-2\pm\sqrt{8}}{-2} = 1\mp\sqrt{2}$

$\Rightarrow \dfrac{2}{2t+1-t^2} = \dfrac{2}{(t-(1+\sqrt{2}))(t-(1-\sqrt{2}))} = \dfrac{A}{t-(1+\sqrt{2})} + \dfrac{B}{t-(1-\sqrt{2})} =$

$= \dfrac{A(t-(1-\sqrt{2}))+B(t-(1+\sqrt{2}))}{2t+1-t^2} \underset{\uparrow}{\Longrightarrow} \left.\begin{array}{l} A+B=0 \\ -A(1-\sqrt{2})-B(1+\sqrt{2})=2 \end{array}\right\}$

összehasonlítva a
számlálókat

$$\Rightarrow \quad B = -A \quad \Rightarrow \quad -A(1-\sqrt{2}) + A(1+\sqrt{2}) = 2$$

$$2\sqrt{2}A = 2 \quad \Rightarrow \quad A = \frac{1}{\sqrt{2}}$$

$$B = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \int \frac{2}{2t+1-t^2}\,dt = \int \frac{\frac{1}{\sqrt{2}}}{t-(1+\sqrt{2})}\,dt - \int \frac{\frac{1}{\sqrt{2}}}{t-(1-\sqrt{2})}\,dt =$$

$$= \frac{1}{\sqrt{2}}\Big(\ln|t-(1+\sqrt{2})| - \ln|t-(1-\sqrt{2})| \Big) + C =$$

$$\underset{\substack{\uparrow \\ t = \operatorname{tg}\frac{x}{2}}}{=} \frac{1}{\sqrt{2}}\Big(\ln\left|\operatorname{tg}\frac{x}{2} - (1+\sqrt{2})\right| - \ln\left|\operatorname{tg}\frac{x}{2} - (1-\sqrt{2})\right| \Big) + C$$

(9.) <u>Racionális törtfüggvények</u>

a) $$\int \frac{1}{x^2-2x+3}\,dx = \int \frac{1}{(x-1)^2+2}\,dx = \frac{1}{2}\int \frac{1}{\left(\frac{x-1}{\sqrt{2}}\right)^2+1}\,dx =$$

$$= \frac{1}{2}\,\frac{\operatorname{arctg}\left(\frac{x-1}{\sqrt{2}}\right)}{1/\sqrt{2}} + C = \underline{\underline{\frac{1}{\sqrt{2}}\operatorname{arctg}\left(\frac{x-1}{\sqrt{2}}\right) + C}}$$

b) $$\int \frac{1}{x^4-x^2}\,dx \underset{\uparrow}{=}$$

$$\frac{1}{x^4-x^2} = \frac{1}{x^2(x^2-1)} = \frac{1}{x^2(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x+1} =$$

$$= \frac{Ax(x-1)(x+1) + B(x-1)(x+1) + Cx^2(x+1) + Dx^2(x-1)}{x^4-x^2}$$

12/ Az ismeretlen együtthatókat meghatározzuk a számlálók összehasonlításával. Ez történhet úgy is, hogy a számlálóba speciálisan behelyettesítünk számokat. Érdemes a gyököket behelyettesíteni:

Pl) a nevező gyökei

$x = 0$ helyettesítésnél $\Rightarrow 1 = -B \Rightarrow B = -1$

$x = 1$ –"– $\Rightarrow 1 = 2C \Rightarrow C = \frac{1}{2}$

$x = -1$ –"– $\Rightarrow 1 = -2D \Rightarrow D = -\frac{1}{2}$

$x = 2$ –"– $\Rightarrow 1 = 6A + 3B + 12C + 4D$

$\hookrightarrow 1 = 6A - 3 + 6 - 2 \Rightarrow A = 0$

$$\Rightarrow \frac{1}{x^4 - x^2} = -\frac{1}{x^2} + \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}$$

$$\hookrightarrow \int \frac{1}{x^4 - x^2}\,dx = -\int \frac{1}{x^2}\,dx + \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{1}{x+1}\,dx =$$

$$= \frac{1}{x} + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

c) $$\int \frac{2x+3}{x^2+3x-10}\,dx \underset{\uparrow}{=} \ln|x^2+3x-10| + C$$

$\cancel{\frac{2x+3}{x^2+3x-10}}$ $(x^2+3x-10)' = 2x+3$

$$\int \frac{f'}{f} = \ln|f| + C$$

d) $\displaystyle\int \frac{x^3-2x^2+4}{x^3(x-2)^2}\,dx$

$$\frac{x^3-2x^2+4}{x^3(x-2)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-2} + \frac{E}{(x-2)^2} =$$

$$= \frac{Ax^2(x-2)^2 + Bx(x-2)^2 + C(x-2)^2 + Dx^3(x-2) + Ex^3}{x^3(x-2)^2}$$

Helyettesítsünk a számlálóba számokat!

- $x=0 \rightsquigarrow 4 = 4C \Rightarrow \boxed{C=1}$
- $x=2 \rightsquigarrow 4 = 8E \Rightarrow \boxed{E=\frac{1}{2}}$
- $x=1 \rightsquigarrow 3 = A+B+C-D+E \rightsquigarrow 3 = A+B+1-D+\frac{1}{2}$

$$A+B-D = \frac{3}{2}$$

- $x=-1 \rightsquigarrow 1 = 9A-9B+9C+3D-E \rightsquigarrow 1 = 9A-9B+9+3D-\frac{1}{2}$

$$9A-9B+3D = -\frac{13}{2}$$

- $x=-2 \rightsquigarrow -12 = 64A-32B+4C+32D-8E$

$$\rightsquigarrow -12 = 64A-32B+4+32D-4$$

$$64A-32B+32D = -12$$

A, B, D

meghatározható

+ kiértékelés ....

13)

e) $\int \frac{\sqrt{x}}{x+1}\,dx \underset{P}{=} \int \frac{t}{t^2+1}\,2t\,dt = 2\int \frac{t^2}{t^2+1}\,dt \ominus$

$t := \sqrt{x} \rightsquigarrow x = t^2 \rightsquigarrow \frac{dx}{dt} = 2t \rightsquigarrow dx = 2t\,dt$

$\ominus\, 2\int \frac{t^2+1-1}{t^2+1}\,dt = 2\int\left(1-\frac{1}{t^2+1}\right)dt = 2\left(t - \operatorname{arctg} t\right) + C \underset{\substack{P \\ t=\sqrt{x}}}{=}$

$= 2\left(\sqrt{x} - \operatorname{arctg}\sqrt{x}\right) + C$

f) $\int \frac{6\,dx}{e^x - 3} \underset{P}{=} \int \frac{6\,\frac{1}{t}\,dt}{t-3} = 6\int \frac{1}{t(t-3)}\,dt \ominus$

$t = e^x \rightsquigarrow x = \ln t \rightsquigarrow \frac{dx}{dt} = \frac{1}{t} \rightsquigarrow dx = \frac{1}{t}\,dt$

$\frac{1}{t(t-3)} = \frac{A}{t} + \frac{B}{t-3} = \frac{A(t-3)+Bt}{t(t-3)}$

- $t = 3 \rightsquigarrow 1 = 3B \Rightarrow B = \frac{1}{3}$
- $t = 0 \rightsquigarrow 1 = -3A \Rightarrow A = -\frac{1}{3}$

$\ominus\, 6\int\left(\frac{-1/3}{t} + \frac{1/3}{t-3}\right)dt = 2\int\left(\frac{1}{t-3} - \frac{1}{t}\right)dt = 2\left(\ln|t-3| - \ln|t|\right) + C =$

$\underset{\substack{P \\ t=e^x}}{=} 2\left(\ln|e^x-3| - \ln e^x\right) + C = 2\ln|e^x-3| - 2x + C$

9) $$\int \frac{4\,dx}{e^{2x}-4} \underset{\uparrow}{=} \int \frac{4}{t^2-4}\,\frac{1}{t}\,dt \;=$$

$$t := e^x \rightsquigarrow x = \ln t \to \frac{dx}{dt} = \frac{1}{t} \Rightarrow dx = \frac{1}{t}\,dt$$

$$\frac{4}{t(t^2-4)} = \frac{4}{t(t-2)(t+2)} = \frac{A}{t} + \frac{B}{t-2} + \frac{C}{t+2} =$$

$$= \frac{A(t-2)(t+2) + Bt(t+2) + Ct(t-2)}{t(t-2)(t+2)}$$

számlálók összehasonlítása:

- $t=0 \rightsquigarrow 4 = -4A \Rightarrow A = -1$
- $t=2 \rightsquigarrow 4 = 8B \Rightarrow B = \frac{1}{2}$
- $t=-2 \rightsquigarrow 4 = -8C \Rightarrow C = -\frac{1}{2}$

$$= \int \left(-\frac{1}{t} + \frac{\frac{1}{2}}{t-2} - \frac{\frac{1}{2}}{t+2}\right)dt = -\ln|t| + \frac{1}{2}\ln|t-2| - \frac{1}{2}\ln|t+2| + c =$$

$$\underset{\substack{\uparrow \\ t=e^x}}{=} -\ln e^x + \frac{1}{2}\ln|e^x-2| - \frac{1}{2}\ln(e^x+2) + c = \underline{\underline{-x + \frac{1}{2}\ln|e^x-2| - \frac{1}{2}\ln(e^x+2) + c}}$$

11)

a) $$\int \frac{1+\sin x}{1-\cos x}\,dx \underset{\uparrow}{=} \int \frac{1+\frac{2t}{1+t^2}}{1-\frac{1-t^2}{1+t^2}} \cdot \frac{2\,dt}{1+t^2} =$$

$$t=\operatorname{tg}\frac{x}{2} \rightsquigarrow \sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$dx = \frac{2\,dt}{1+t^2}$$

$$= \int \frac{1+t^2+2t}{1+t^2-(1-t^2)}\,\frac{2}{1+t^2}\,dt = \int \frac{t^2+2t+1}{2t^2}\,\frac{2}{1+t^2}\,dt =$$

$$= \int \frac{t^2+2t+1}{t^2(1+t^2)}\,dt \;\; (=)$$

$$\frac{t^2+2t+1}{t^2(1+t^2)} = \frac{A}{t} + \frac{B}{t^2} + \frac{Ct+D}{1+t^2} = \frac{At(1+t^2)+B(1+t^2)+(Ct+D)t^2}{t^2(1+t^2)} =$$

$$= \frac{t^3(A+C)+t^2(B+D)+t(A)+B}{t^2(1+t^2)} \qquad \forall\, t\in\mathbb{R}\setminus\{0\}$$

együtthatók összehasonlítása:

$$\left.\begin{array}{lll} t^3 & : & A+C=0 \\ t^2 & : & B+D=1 \\ t & : & A=2 \\ t^0 & : & B=1 \end{array}\right\} \quad A=2,\ B=1,\ C=-2,\ D=0$$

$$(=) \int\left(\frac{2}{t}+\frac{1}{t^2}-\frac{2t}{1+t^2}\right)dt = 2\ln|t| - \frac{1}{t} - \ln(1+t^2) + C \underset{t=\operatorname{tg}\frac{x}{2}}{\underset{\uparrow}{=}}$$

$$= 2\ln\left|\operatorname{tg}\frac{x}{2}\right| - \operatorname{cotg}\frac{x}{2} - \ln\left(1+\operatorname{tg}^2\frac{x}{2}\right) + C$$

i)

$$\int x\sqrt{5x+3}\,dx \underset{P}{=} \int \frac{t^2-3}{5}\cdot t\,\frac{2t}{5}\,dt \;(=)$$

$$t := \sqrt{5x+3} \rightsquigarrow x = \frac{t^2-3}{5} \rightsquigarrow \frac{dx}{dt} = \frac{2t}{5}$$

$$\rightsquigarrow dx = \frac{2t}{5}\,dt$$

$$(=)\; \frac{2}{25}\int (t^4 - 3t^2)\,dt = \frac{2}{25}\left(\frac{t^5}{5} - t^3\right) + C \underset{P}{=}$$

$$t = \sqrt{5x+3}$$

$$= \frac{2}{25}\left(\frac{\sqrt{(5x+3)^5}}{5} - \sqrt{(5x+3)^3}\right) + C$$

15)

Határozott integrál alkalmazásai

① Integrálfüggvény

a)
$$\lim_{x\to 0}\frac{\int_0^x t^2\arcsin t\,dt}{x}\underset{L}{=}\lim_{x\to 0}\frac{\overbrace{x^2\arcsin x}^{\to 0}}{1}=0$$

$\frac{0}{0}$ típusú ⇝ l'Hospital

$f(t)=t^2\arcsin t$ folytonos ezért az integrálfv deriválható, ha $x<1$.

(arcsin értelmezése miatt)

b)
$$\lim_{x\to 0}\frac{\int_0^x \cos t^2\,dt}{x}\underset{L}{=}\lim_{x\to 0}\frac{\cos x^2}{1}=\cos 0=\underline{\underline{1}}$$

$\frac{0}{0}$ ⇝ l'Hospital

$\cos t^2$ folytonos ⇝ integrálfv deriválható

c)
$$\lim_{x\to\infty}\frac{\left(\int_0^x e^{t^2}dt\right)^2}{\int_0^x e^{2t^2}dt}\underset{L}{=}\lim_{x\to\infty}\frac{2\left(\int_0^x e^{t^2}dt\right)\cdot e^{x^2}}{e^{2x^2}}=$$

$\frac{\infty}{\infty}$ ⇝ l'Hospital

e^{t^2} és e^{2t^2} folytonos

$$=\lim_{x\to\infty}\frac{2\int_0^x e^{t^2}dt}{e^{x^2}}\underset{L}{=}\lim_{x\to\infty}\frac{2e^{x^2}}{e^{x^2}\cdot 2x}=\lim_{x\to\infty}\frac{1}{x}=\underline{\underline{0}}$$

$\frac{\infty}{\infty}$ ⇝ l'Hospital

d) $\frac{d}{dx}\int_x^{2x} \ln(1+t)\,dt = ?$ $(x>0)$

$F(x) := \int_0^x \ln(1+t)\,dt \rightsquigarrow F'(x) = \ln(1+x)$

$\ln(1+t)$ folytonos $\rightsquigarrow F(x)$ deriválható

$G(x) := \int_0^{2x} \ln(1+t)\,dt = F(2x) \rightsquigarrow G'(x) = F'(2x)\cdot 2 = 2\ln(1+2x)$

$H(x) := \int_x^{2x} \ln(1+t)\,dt = \int_0^{2x} \ln(1+t)\,dt - \int_0^x \ln(1+t)\,dt =$

$= G(x) - F(x) \rightsquigarrow H'(x) = G'(x) - F'(x) =$

$= \underline{2\ln(1+2x) - \ln(1+x)}$

$\cancel{= \ln(1+x)}$

e) $\frac{d}{dx}\int_{x^2}^{x^3} \frac{dt}{\sqrt{1+t^4}} = ?$

$F(x) := \int_0^x \frac{dt}{\sqrt{1+t^4}}$ deriválható, mert $\frac{1}{\sqrt{1+t^4}}$ folytonos

$\rightsquigarrow F'(x) = \frac{1}{\sqrt{1+x^4}}$

$G(x) := \int_0^{x^3} \frac{1}{\sqrt{1+t^4}}\,dt = F(x^3) \rightsquigarrow G'(x) = F'(x^3)\cdot 3x^2 = \frac{3x^2}{\sqrt{1+x^{12}}}$

$H(x) := \int_{x^2}^{x^3} \frac{dt}{\sqrt{1+t^4}} = \int_0^{x^3} \frac{dt}{\sqrt{1+t^4}} - \int_0^{x^2} \frac{dt}{\sqrt{1+t^4}} = F(x^3) - F(x^2) = G(x) - F(x^2)$

$\rightsquigarrow H'(x) = G'(x) - F'(x^2)\cdot 2x = \frac{3x^2}{\sqrt{1+x^{12}}} - \frac{2x}{\sqrt{1+x^8}}$