Calculus 1, Practise Course

1st week

I. Warming up

1. Solve the following inequalities.
   (a) $|2x-3|<1$
   (b) $x^2+2x-8\leq 0$
   (c) $(x-2)^2\geq 0$
   (d) $|x^2-7x+12|>x^2-7x+12$
   (e) $|3x-5|-|2x+3|>0$
   (f) $|x^2-5x|>|x^2|-|5x|$
2. Find out whether the following equations have any solutions.
   (a) $|x|=x+5$
   (b) $|x|=x-5$
3. Determine the values of x satisfying the following equalities.
   (a) $\left|\frac{x-1}{x+1}\right|=\frac{x-1}{x+1}$
   (b) $|x^2-5x+6|=-(x^2-5x+6)$
   (c) $|(x^2+4x+9)+(2x-3)|=|x^2+4x+9|+|2x-3|$
   (d) $|(x^4-4)-(x^2+2)|=|x^4-4|-|x^2+2|$
4. Find the roots of the following equations.
   (a) $|\sin x|=\sin x+1$
   (b) $x^2-2|x|-3=0$
5. Given the function $f(x)=\frac{x+1}{x-1}$, $(x\neq 1)$. Find $f(2x), 2f(x), f(x^2), [f(x)]^2$.

6. Find $f(x)$ if $f(x+1) = x^2 - 3x + 2$.
7. Given the function $f(x) = \frac{5x^2+1}{2-x}$ find $f(3x), f(x^3), 3f(x), [f(x)]^3$.
8. Given the function $f(x) = \ln \frac{1-x}{1+x}$. Find the domain of f. Show that at $x_1, x_2 \in (-1, 1)$ the following identity holds true:
$$f(x_1) + f(x_2) = f\left(\frac{x_1 + x_2}{1 + x_1 \cdot x_2}\right).$$
9. Given the function $f(x) = \frac{a^x + a^{-x}}{2}$, $a > 0$. Show that
$$f(x+y) + f(x-y) = 2f(x)f(y).$$
10. Find a function of the form $f(x) = a + bc^x$, $c > 0$ if $f(0) = 15, f(2) = 30, f(4) = 90$.

II. Domains and ranges

1. Find the domains of the following functions.
   (a) $f(x) = \sqrt{1-x^2}$
   (b) $f(x) = \sqrt{1-\sqrt{1-x^2}}$
   (c) $f(x) = \frac{1}{x-1} + \frac{1}{x-2}$
   (d) $f(x) = \sqrt{1-x^2} + \sqrt{x^2-1}$
   (e) $f(x) = \sqrt{1-x} + \sqrt{x-2}$
   (f) $f(x) = \log_2 \log_3 \log_4 x$
   (g) $f(x) = \ln|4 - x^2|$
   (h) $f(x) = \frac{1}{\ln(1+x)}$
   (i) $f(x) = \frac{2x-3}{\sqrt{x^2+2x+3}}$
   (j) $f(x) = \log_2 \sin(x-3) + \sqrt{16-x^2}$
2. Find the domains and the ranges of the following functions.
   (a) $f(x) = \sqrt{\cos(\sin x)}$
   (b) $f(x) = \frac{1}{2-\cos x}$
   (c) $f(x) = \frac{1}{2-\cos 3x}$
   (d) $f(x) = \frac{x}{1+x^2}$

II. Linear functions

1. Find and graph the linear function

   (a) that passes through the points $(1, 3)$ and $(2, 5)$

   (b) that passes through the points $(2, -3)$ and $(5, 0)$

   (c) that passes through the point $(3, 2)$ and is parallel to the line $y = 3x + 8$

   (d) that passes through the points $(-1, 4)$ and perpendicular to the line $y = \frac{x}{4} - 7$

   (e) that passes through the points $(1, 3)$ and its slope is $m = -2$

   (f) that has y-intercept -3 and slope $m = 1/3$

2. Converting Celsius temperature (C) to Fahrenheit temperature (F) is a linear function. Find and graph this $F(C)$ linear function, if we know that $F = 32$ if $C = 0$ and $F = 212$ if $C = 100$. What is the $C(F)$ function? Is there a temperature at which a Fahrenheit thermometer gives the same reading as a Celsius thermometer? If so, what is it?

3. A ray of light comes in along the line $x + y = 1$ above the x-axis and reflects off the x-axis. The angle of departure is equal to the angle of arrival. Write an equation for the line along which the departing light travels.

III. Some properties of functions

1. Find the intervals of increase and decrease of the function $f(x) = ax^2 + bx + c$, and its minimum and maximum values. Apply your results to find the rectangle with the maximum area from among all rectangles of a given perimeter.

2. Let consider the function

   $$f(x) = a \cos x + b \sin x \quad (a^2 + b^2 > 0).$$

   Show that the given function can be represented as

   $$f(x) = \sqrt{a^2 + b^2} \cos(x - \alpha),$$

   where $\cos \alpha = a/\sqrt{a^2 + b^2}$ and $\sin \alpha = b/\sqrt{a^2 + b^2}$. Find the minimum and the maximum values of the function f. With the help of the expression above, give the intervals of increase and decrease for the function

   $$g(x) = \cos x + \sin x.$$

3. Show that

(a) the function $f(x) = x^3 + 3x + 5$ increases in the entire domain (don't use derivation!).

(b) the function $g(x) = \frac{x}{1+x^2}$ decreases in the interval $(1, \infty)$ (don't use derivation!).

4. Find the minimum value of the function

$$f(x) = 3^{(x^2-2)^3+8}.$$

5. Decide whether the following function is even, odd or neither one.

   (a) $f(x) = \log_3(x + \sqrt{1 + x^2}$

   (b) $f(x) = \ln \frac{1-x}{1+x}$

   (c) $f(x) = 2x^3 - x + 1$

   (d) $f(x) = 4 - 2x^4 + \sin^2 x$

   (e) $f(x) = \sqrt{1 + x + x^2} - \sqrt{1 - x + x^2}$

6. Prove that if $f(x)$ is a periodic function with period T, then the function $f(ax+b)$, where $a > 0$, is periodic with period T/a.

7. The periodic function

$$f(x) = A\sin(\omega x + \varphi)$$

is called a *harmonic function* with amplitude $|A|$, frequency ω and initial phase φ. From the problem above, we know that $f(x)$ is periodic with period $T = 2\pi/\omega$. Indicate the amplitude $|A|$, frequency ω, initial phase φ and period T of the following harmonics:

   (a) $f(x) = 3\sin(x/2) + 4\cos(x/2)$

   (b) $f(x) = 4\sin 2x \cos 2x$

8. Find the period for each of the following functions:

   (a) $f(x) = \tan 2x$

   (b) $f(x) = \sin 2\pi x$

   (c) $f(x) = \sin^4 x + \cos^4 x$

   (d) $f(x) = |\cos x|$

9. Prove that the function $f(x) = \cos x^2$ is not a periodic one.