Calculus 1, Practise Course

2nd week

I. Composite functions

1. Let $f(x) = x^2$, $g(x) = 2^x$ and $h(x) = \sin x$. Determine the following functions
   (a) $(f \circ g)(x)$
   (b) $(f \circ h)(x)$
   (c) $(f \circ g \circ h)(t) + (h \circ g)(t)$
2. Express each of the following functions in terms of f, g, h (see above) using only the operations $+, \cdot, \circ$.
   (a) $F(x) = 2^{\sin x}$
   (b) $F(x) = \sin 2^x$
   (c) $F(x) = \sin^2 x$
   (d) $F(t) = 2^{2^t}$ (a^{b^c} always means $a^{(b^c)}$, because $(a^b)^c = a^{b \cdot c}$ in simpler form.)
   (e) $F(u) = \sin(2^u + 2^{u^2})$
   (f) $F(y) = \sin(\sin(\sin(2^{2^{2^{\sin y}}})))$
   (g) $F(x) = 2^{\sin^2 x} + \sin x^2 + 2^{\sin(x^2 + \sin x)}$
3. Let $g(x) = x^2 + 3$. Find a function f that produces the given decomposition.
   (a) $(f \circ g)(x) = x^2$
   (b) $(f \circ g)(x) = x^4 + 6x^2 + 9$
   (c) $(g \circ f)(x) = x^4 + 3$
   (d) $(g \circ f)(x) = x^{2/3} + 3$
   (e) $(f \circ g)(x) = \frac{1}{x^2+3}$

4. Let $g(x) = x^2$ and let

$$h(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1. & \text{if } x \text{ is irrational.} \end{cases}$$

   (a) For which y is $h(y) \leq y$?
   (b) For which y is $h(y) \leq g(y)$?
   (c) What is $g(h(x)) - h(x)$?
   (d) For which w is $g(w) \leq w$?

5. For which number a, b, c and d will the function

$$f(x) = \frac{ax + b}{cx + d}$$

   satisfy that $(f \circ f)(x) = x$ for all real x?

6. Let consider the following functions.

$$f(x) = \begin{cases} 2x - 1, & \text{if } x \in (-\infty, 1] \\ 2, & \text{if } x \in [1, 5] \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \frac{6-x}{7-x}, & \text{if } x \in (-\infty, 6) \\ \left(1 + \frac{1}{x}\right)^6, & \text{if } x \in [6, \infty). \end{cases}$$

   Give $f \circ g$.

II. Inverse functions

1. Find the inverse of the following functions

   (a) $f(x) = \frac{x+1}{x-2}$, $x \neq 2$.
   (b) $f(x) = (1 - x^3)^{1/5} + 2$
   (c) $f(x) = x^3 + 6x^2 + 12x$, $x \in \mathbb{R}$.
   (d) $f(x) = \frac{e^x}{e^x+2}$
   (e) $f(x) = \frac{x}{x-2}$, i) for $x > 2$, ii) for $x < 2$
   (f) $f(x) = \frac{1}{2x+3}$, $D_f = \mathbb{R} \setminus \{-\frac{3}{2}\}$.
   (g) $f(x) = \left(\frac{x-1}{1+x}\right)^2 - 1$, $(x \in (-1, 1))$
   (h) $f(x) = x^2$, $D_f = (-\infty, -1]$.
   (i) $f(x) = x^3 - 3x^2 + 3x + 4$

(j)

$$f(x) = \begin{cases} \frac{7x-5}{3}, & \text{if } -1 \le x < 1 \\ \frac{2}{1+x}, & \text{if } 1 \le x \le 2. \end{cases}$$

(k) $f(x) = \log_a(x + \sqrt{x^2+1})$, $(a > 1, a \neq 1)$

2. For which real α numbers will be the following function invertible? Give the inverse function, including its domain and its range.

$$f(x) = \begin{cases} \alpha x^2, & \text{if } -1 \le x < 0 \\ 2\alpha - x, & \text{if } 0 < x \le 1. \end{cases}$$

3. Show that the functions

$$f(x) = x^2 - x + 1, \quad (x \ge 1/2) \quad \text{and} \quad \varphi(x) = 1/2 + \sqrt{x - 3/4}$$

are mutually inverse and with this knowledge solve the equation

$$x^2 - x + 1 = 1/2 + \sqrt{x - 3/4}.$$

III. Transformations of functions and graphs

1. Sketch the graph of the following functions

(a) $f(x) = -\sqrt{2x+1}$

(b) $f(x) = \sqrt{1 - x/2}$

(c) $f(x) = (x-1)^3 + 2$

(d) $f(x) = (1-x)^3 + 2$

(e) $f(x) = |x^2 - 1|$

(f) $f(x) = \frac{1}{2x} - 1$

(g) $f(x) = 3\sqrt{-2(x+5/2)} - 4/5$

(h) $f(x) = \frac{x+3}{x+1}$

(i) $f(x) = 3\cos x - \sqrt{3}\sin x$
Hint: transform the given function to $f(x) = A\cos(x + \phi)$ form.

2. Graph both f and f^{-1} on the same set of axes.

(a) $f(x) = \sqrt{x+2}$, $(x \ge -2)$

(b) $f(x) = \sqrt{3-x}$, $(x \leq 3)$

(c) $f(x) = (x-2)^2 - 1$, for $x \geq 2$

(d) $f(x) = e^{2x+6}$

IV. Symmetry of functions and graphs

1. Determine whether the graphs of the following equations and functions are symmetric about the x-axis, the y-axis or the origin.

   (a) $f(x) = x^4 - 5x^2 - 12$

   (b) $f(x) = 3x^7 - 5x^5 - 5x$

   (c) $f(x) = x^5 - x^3 + 3$

   (d) $x^{2/3} + y^{2/3} = 1$

   (e) $x^3 - y^5 = 0$

   (f) $|x| + |y| = 1$

2. Assume f is an even function and g is an odd function, the domain of both of them is the all real line. What can we say about the symmetry of the following functions?

   (a) $f \cdot g$

   (b) f/g

   (c) $f \circ g$

   (d) $f \circ f$

   (e) $g \circ g$

   (f) f^2

   (g) g^2